EVA

Extreme Value Analysis

Technical Reference and Documentation

Please Note

Copyright

Limited Liability

The liability of DHI is limited as specified in Section III of your 'DHI Software Licence Agreement':

'IN NO EVENT SHALL DHI OR ITS REPRESENTATIVES (AGENTS AND SUPPLIERS) BE LIABLE FOR ANY DAMAGES WHATSOEVER INCLUDING, WITHOUT LIMITATION, SPECIAL, INDIRECT, INCIDENTAL OR CONSEQUENTIAL DAMAGES OR DAMAGES FOR LOSS OF BUSINESS PROFITS OR SAVINGS, BUSINESS INTERRUPTION, LOSS OF BUSINESS INFORMATION OR OTHER PECUNIARY LOSS ARISING OUT OF THE USE OF OR THE INABILITY TO USE THIS DHI SOFTWARE PRODUCT, EVEN IF DHI HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES. THIS LIMITATION SHALL APPLY TO CLAIMS OF PERSONAL INJURY TO THE EXTENT PERMITTED BY LAW. SOME COUNTRIES OR STATES DO NOT ALLOW THE EXCLUSION OR LIMITATION OF LIABILITY FOR CONSEQUENTIAL, SPECIAL, INDIRECT, INCIDENTAL DAMAGES AND, ACCORDINGLY, SOME PORTIONS OF THESE LIMITATIONS MAY NOT APPLY TO YOU. BY YOUR OPENING OF THIS SEALED PACKAGE OR INSTALLING OR USING THE SOFTWARE, YOU HAVE ACCEPTED THAT THE ABOVE LIMITATIONS OR THE MAXIMUM LEGALLY APPLICABLE SUBSET OF THESE LIMITATIONS APPLY TO YOUR PURCHASE OF THIS SOFTWARE.'

Printing History

June 2005
April 2006
October 2007

CONTENTS

1 INTRODUCTION

The EVA toolbox in MIKE Zero comprises a comprehensive suite of routines for performing extreme value analysis. These include

- A pre-processing facility for extraction of the extreme value series from the record of observations.
- Support of two different extreme value models, the annual maximum series model and the partial duration series model.
- Support of a large number of probability distributions, including exponential, generalised Pareto, Gumbel, generalised extreme value, Weibull, Frechét, gamma, Pearson Type 3, Log-Pearson Type 3, log-normal, and square-root exponential distributions.
- Three different estimation methods: method of moments, maximum likelihood method, and method of L-moments.
- Three validation tests for independence and homogeneity of the extreme value series.
- Calculation of five different goodness-of-fit statistics.
- Support of two different methods for uncertainty analysis, Monte Carlo simulation and Jackknife resampling.
- Comprehensive graphical tools, including histogram and probability plots.

This document provides a technical reference and documentation for the different tools available in EVA.

2 EXTREME VALUE MODELS

For evaluating the risk of extreme events a parametric frequency analysis approach is adopted in EVA. This implies that an extreme value model is formulated based on fitting a theoretical probability distribution to the observed extreme value series. Two different extreme value models are provided in EVA, the annual maximum series (AMS) method and the partial duration series (PDS) method, also known as the peak over threshold (POT) method.

2.1 Basic probabilistic concepts

The defined extreme value population is described by a stochastic variable X. The cumulative distribution function $F(x)$ is the probability that X is less than or equal to x

$$F(x) = P\{X \le x\} \tag{2.1}$$

The probability density function $f(x)$ for a continuous random variable is defined as the derivative of the cumulative distribution function

$$f(x) = \frac{dF(x)}{dx} \tag{2.2}$$

The quantile of a distribution is defined as

$$x_p = F^{-1}(p) \tag{2.3}$$

where $p = P\{X \le x\}$. The quantile x_p is exceeded with probability (1-p), and hence is often referred to as the (1-p)-exceedance event. Often the return period of the event is specified rather than the exceedance probability. If (1-p) denotes the exceedance probability in a year, the return period T is defined as

$$T = \frac{1}{1-p} \tag{2.4}$$

Correspondingly, the T-year event x_T calculated from (2.3) is the level, which on the average is exceeded once in T years.

2.2 Annual maximum series

In the annual maximum series (AMS) method the maximum value in each year of the record are extracted for the extreme value analysis (see Figure 2.1). The analysis year should preferably be defined from a period of the year where extreme events never or very seldomly occur in order to ensure that a season with extreme events is not split in two. Alternatively, a specific season may be defined as the analysis year.

For estimation of T-year events, a probability distribution $F(x)$ is fitted to the extracted AMS data $\{x_i, i = 1,2,\ldots,n\}$ where n is the number of years of record. The T-year event estimate is given by

$$\hat{x}_T = F^{-1}\left(1 - \frac{1}{T}; \hat{\theta}\right) \tag{2.5}$$

where $\hat{\theta}$ are the estimated distribution parameters.

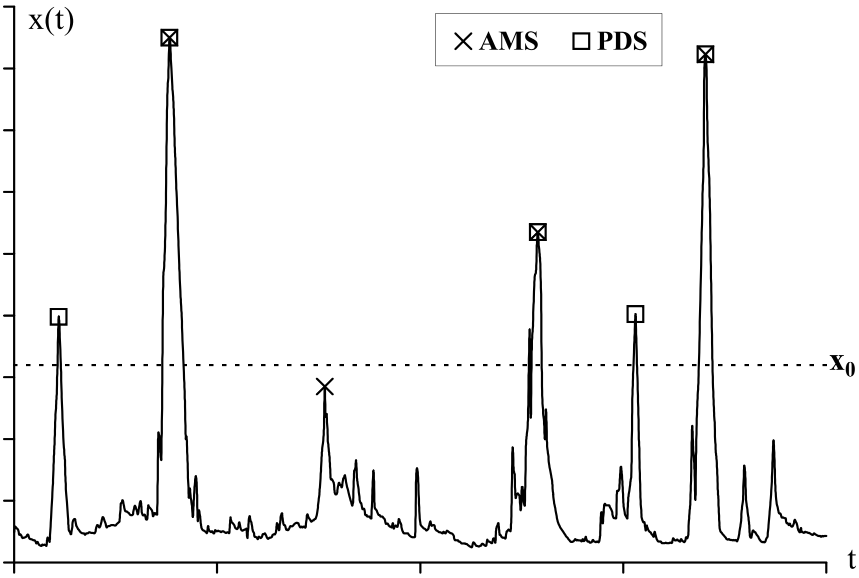

Figure 2.1 Extraction of AMS and PDS from the recorded time series.

2.3 Partial duration series

In the partial duration series (PDS) method all events above a threshold are extracted from the time series (see Figure 2.1). The PDS can be defined in two different ways. In Type I sampling, all events above a predefined threshold x_0 are taken into account $\{x_i > x_0, i = 1,2,\ldots,n\}$, implying that the number of exceedances n becomes a random variable. In Type II sampling, the n largest events are extracted $\{x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}\}$, implying that the threshold level becomes a random variable. If n equals the number of observation years, the PDS is referred to as the annual exceedance series.

In EVA, both the Type I and Type II sampling methods are provided as pre-processing tools for extracting the PDS. If Type I sampling (fixed threshold level) is chosen, the corresponding number of exceedances is calculated. Similarly, if Type II sampling is chosen (fixed number of events or, equivalently, fixed average annual number of events), the corresponding threshold level is determined. For definition of the PDS both the threshold level and the average annual number of events have to be specified.

To ensure independent events in the PDS, usually some restrictions have to be imposed on the time and level between two successive events. In EVA, an interevent time and interevent level criterion can be defined:

1 Interevent time criterion Δt_c: two successive events are independent if the time between the two events is larger than Δt_c.

2 Interevent level criterion p_c $(0 < p_c < 1)$: two successive events are independent if the level between the events becomes smaller than p_c times the lower of the two events.

If both criteria are chosen, two successive events are independent only if both (1) and (2) are fulfilled.

If a fixed threshold level is used to define the extreme value series (Type I sampling), the PDS model includes two stochastic modelling components, respectively, the occurrence of extreme events and the exceedance magnitudes. It is assumed that the occurrence of exceedances can be described by a Poisson process with constant or one-year periodic intensity, implying that the number of exceedances n is Poisson distributed with probability function

$$P\{N(t) = n\} = \frac{(\lambda t)^n}{n!}\exp(-\lambda t) \tag{2.6}$$

where t is the recording period. The Poisson parameter λ equals the expected number of exceedances per year and is estimated from the record as

$$\hat{\lambda} = \frac{n}{t} \tag{2.7}$$

For modelling the exceedance magnitudes a probability distribution $F(x-x_0)$ is fitted to the exceedance series $\{x_i-x_0,\ i = 1,2,\ldots,n\}$. The T-year event estimate is given by

$$\hat{x}_T = x_0 + F^{-1}\left(1-\frac{1}{\hat{\lambda}T};\hat{\theta}\right) \tag{2.8}$$

where $\hat{\theta}$ are the estimated distribution parameters.

In the case of Type II sampling, the average annual number of events λ is fixed. For modelling the extremes a probability distribution $F(x)$ is fitted to the extreme value series $\{x_i,\ i = 1,2,\ldots,n\}$. The T-year event estimate is given by

$$\hat{x}_T = F^{-1}\left(1-\frac{1}{\lambda T};\hat{\theta}\right) \tag{2.9}$$

where $\hat{\theta}$ are the estimated distribution parameters.

The T-year event in the PDS can also be related to the return period of the corresponding annual maximum series (denoted annual return period T_A). The relationship between the return period T defined above and T_A is given by

$$\frac{1}{T_A} = 1 - \exp\left(-\frac{1}{T}\right) \tag{2.10}$$

Note that for return periods larger than about 10 years T and T_A are virtually identical.

3 INDEPENDENCE AND HOMOGENEITY TESTS

The basic requirements for the extreme value models outlined above is that the stochastic variables X_i are independent and identically distributed. For testing independence and homogeneity of the observed extreme value series, three different tests are available in EVA

- Run test
- Mann-Kendall test
- Mann-Whitney test

3.1 Run test

The run test is used for general testing of independence and homogeneity of a time series. From the time series $\{x_i, i = 1,2,\ldots,n\}$ the sample median x_{med} is calculated and a shifted series $\{s_i = x_i - x_{med}, i = 1,2,\ldots,n\}$ is constructed. From the shifted series a run is defined as a set of successive elements having the same sign. The test statistic is given as the number of runs of the shifted series, i.e.

$$z = \sum_{i=2}^{n} \mathrm{sgn}(i) \qquad , \qquad \mathrm{sgn}(i) = \begin{cases} 1 & , s_{i-1}s_i < 0 \\ 0 & , s_{i-1}s_i > 0 \end{cases} \tag{3.1}$$

The test statistic is asymptotically normally distributed with mean μ_z and variance σ_z^2 given by

$$\begin{aligned} \mu_z &= \frac{n}{2} + 1 \\ \sigma_z^2 &= \frac{n(n-2)}{4(n-1)} \end{aligned} \tag{3.2}$$

Thus, the standardised test statistic

$$z^* = \begin{cases} \dfrac{z - \mu_z - 1/2}{\sigma_z} & , z > \mu_z \\ 0 & , z = \mu_z \\ \dfrac{z - \mu_z + 1/2}{\sigma_z} & , z < \mu_z \end{cases} \tag{3.3}$$

is evaluated against the quantiles of a standard normal distribution. That is, the H_0 hypothesis of independent and homogeneous data is rejected at significance level α if $|z^*| > \Phi^{-1}(1-\alpha/2)$ where $\Phi^{-1}(1-\alpha/2)$ is the $(1-\alpha/2)$-quantile in the standard normal distribution.

3.2 Mann-Kendall test

The Mann-Kendall test is used for testing monotonic trend of a time series $\{x_i, i = 1,2,..,n\}$. The test statistic reads

$$z = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{sgn}(x_j - x_i) \tag{3.4}$$

where

$$\mathrm{sgn}(x_j - x_i) = \begin{cases} 1 & , x_j > x_i \\ 0 & , x_j = x_i \\ -1 & , x_j < x_i \end{cases} \tag{3.5}$$

A positive value of z indicates an upward trend, whereas a negative value indicates a downward trend. The test statistic is asymptotically normally distributed with zero mean ($\mu_z = 0$) and variance given by

$$\sigma_z^2 = \frac{1}{18} n(n-1)(2n+5) \tag{3.6}$$

For evaluating the H_0 hypothesis: no trend in the series, the standardised test statistic calculated from (3.3) is compared to the quantiles of a standard normal distribution.

3.3 Mann-Whitney test

The Mann-Whitney test is used for testing shift in the mean between two sub-samples defined from a time series $\{x_i, i = 1,2,..,n\}$. For the time series ranks R_i are assigned from $R_i = 1$ for the smallest to $R_i = n$ for the largest observation. Time series of ranks for the two-subsamples are then defined by $\{R_i, i = 1,2,..,n_1\}$ and $\{R_i, i = 1,2,..,n_2\}$ where $n = n_1 + n_2$. The test statistic is given as the sum of ranks of the smaller sub-series, i.e.

$$z = \sum_{i=1}^{m} R_i \qquad , \quad m = Min\{n_1, n_2\} \tag{3.7}$$

The test statistic is asymptotically normally distributed with mean and variance

$$\begin{aligned} \mu_z &= \frac{m(n+1)}{2} \\ \sigma_z^2 &= \frac{n_1 n_2 (n+1)}{12} \end{aligned} \tag{3.8}$$

For evaluating the H_0 hypothesis: same mean value in the two sub-series, the standardised test statistic calculated from (3.3) is compared to the quantiles of a standard normal distribution.

4 PROBABILITY DISTRIBUTIONS

4.1 Probability distribution for AMS

The probability distributions that can be applied for AMS are shown in Table 4.1. The probability density function, the cumulative distribution function, and the quantile function for these distributions are given in Appendiks A.

Table 4.1 Combinations of probability distributions and estimation methods (method of moments (MOM), L-moments (LMOM), and maximum likelihood (ML)) available for AMS.

Distribution	No. of parameters	MOM	LMOM	ML
Gumbel	2	x	x	x
Generalised extreme value	3	x	x	x
Weibull	3	x	x	
Frechét	3	x		
Generalised Pareto	3	x	x	
Gamma/Pearson Type 3	3	x	x	
Log-Pearson Type 3	3	x	x	
Log-normal	2	x	x	x
	3	x		x
Square root exponential	2			x

For the log-normal distribution both a 2- and a 3-parameter version is available. In the 2-parameter version the location parameter is set equal to zero.

4.2 Probability distributions for PDS

The probability distributions that can be applied for PDS are shown in Table 4.2. The probability density function, the cumulative distribution function, and the quantile function for these distributions are given in Appendix A.

If the PDS is defined using a fixed threshold, the location parameter is set equal to the threshold level x_0, and the remaining distribution parameters are estimated from the exceedance series $\{x_i - x_0,\ i = 1,2,\ldots,n\}$. On the other hand, when the PDS is defined using a fixed average annual number of events, the location parameter is estimated from the data $\{x_i,\ i = 1,2,\ldots,n\}$ along with the other distribution parameters. The three parameters of the log-Pearson Type 3 distribution and the two parameters of the truncated Gumbel distribution are estimated from the data $\{x_i,\ i = 1,2,\ldots,n\}$.

Table 4.2 Combinations of probability distributions and estimation methods (method of moments (MOM), L-moments (LMOM), and maximum likelihood (ML)) available for PDS.

Distribution	**Location parameter**	**No. of parameters**	**MOM**	**LMOM**	**ML**
Exponential	Fixed	1	x	x	x
	Estimated	2	x	x	
Generalised Pareto	Fixed	2	x	x	x
	Estimated	3	x	x	
Weibull	Fixed	2	x	x	x
	Estimated	3	x	x	
Gamma/Pearson Type 3	Fixed	2	x	x	x
	Estimated	3	x	x	
Log-normal	Fixed	2	x	x	x
	Estimated	3	x		x
Log-Pearson Type 3	-	3	x	x	
Truncated Gumbel	-	2			x

5 ESTIMATION METHODS

For estimation of the parameters of the probability distributions three different estimation methods are available

- Method of moments
- Method of L-moments
- Maximum likelihood method

The estimation methods that are available for the different distributions are shown in Table 4.1 and Table 4.2.

5.1 Method of moments

The product moments: mean value μ, variance σ^2, coefficient of skewness γ_3, and kurtosis γ_4 are defined as

$$\begin{aligned}
\mu &= E\{X\} \\
\sigma^2 &= Var\{X\} = E\{(X-\mu)^2\} \\
\gamma_3 &= \frac{E\{(X-\mu)^3\}}{\sigma^3} \\
\gamma_4 &= \frac{E\{(X-\mu)^4\}}{\sigma^4}
\end{aligned} \tag{5.1}$$

where $E\{.\}$ is the expectation operator. The standard deviation σ is the square root of the variance. Population moments for the distributions available in EVA are shown in Appendix A.

Based on the set of observations $\{x_i, i = 1,2,\ldots,n\}$, estimators of the product moments can be calculated

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{5.2}$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \tag{5.3}$$

$$\hat{\gamma}_3 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2\right]^{3/2}} \tag{5.4}$$

$$\hat{\gamma}_4 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^4}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2\right]^{2}} \tag{5.5}$$

The moment estimators of the distribution parameters are then obtained by replacing the theoretical product moments for the specified distribution by the sample moments. Expressions of the moment estimators for the different distributions are given in Appendix A.

5.2 Method of L-moments

L-moments are defined as linear combinations of expected values of order statistics [Hosking, 1990]. The first L-moment (λ_1) is the mean value identical to the first ordinary moment. The second L-moment (λ_2) is a measure of scale or dispersion analogous to standard deviation, and the third (λ_3) and fourth (λ_4) L-moments are measures of skewness and kurtosis, respectively. L-moments can be written as linear combinations of probability weighted moments (PWM). The PWM of order r is defined as

$$\beta_r = E\{XF(X)^r\}, r = 1, 2, \ldots \tag{5.6}$$

The first four L-moments in terms of PWMs read

$$
\begin{aligned}
\lambda_1 &= \beta_0 \\
\lambda_2 &= 2\beta_1 - \beta_0 \\
\lambda_3 &= 6\beta_2 - 6\beta_1 + \beta \\
\lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0
\end{aligned}
\tag{5.7}
$$

Analogous to the skewness and kurtosis defined by product moments, the L-skewness (τ_3) and L-kurtosis (τ_4) are defined as

$$
\tau_3 = \frac{\lambda_3}{\lambda_2} \quad , \quad \tau_4 = \frac{\lambda_4}{\lambda_2}
\tag{5.8}
$$

Since the first r L-moments can be expressed in terms of the first r PWMs, procedures based on L-moments and PWM are similar. L-moments, however, are more convenient with respect to summarising a probability distribution. Population L-moments for the distributions available in EVA are shown in Appendix A.

For estimation of L-moments, unbiased PWM estimators are employed [Landwehr et al., 1979]

$$
\begin{aligned}
\hat{\beta}_0 &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
\hat{\beta}_1 &= \frac{1}{n}\sum_{i=1}^{n-1} \frac{n-i}{n-1} x_{(i)} \\
\hat{\beta}_2 &= \frac{1}{n}\sum_{i=1}^{n-2} \frac{(n-i)(n-i-1)}{(n-1)(n-2)} x_{(i)} \\
\hat{\beta}_3 &= \frac{1}{n}\sum_{i=1}^{n-3} \frac{(n-i)(n-i-1)(n-i-2)}{(n-1)(n-2)(n-3)} x_{(i)}
\end{aligned}
\tag{5.9}
$$

where $x_{(n)} \leq x_{(n-1)} \leq \ldots \leq x_{(1)}$ is the ordered sample of observations. Unbiased L-moment estimators are obtained by replacing the PWMs in (5.7) by their sample estimates in (5.9). L-moment estimates of the distribution parameters are then obtained by replacing the theoretical L-moments for the specified distribution by the L-moment estimators. Expressions of the L-moment estimators for the different distributions are given in Appendix A.

5.3 Maximum likelihood method

Maximum likelihood estimators are obtained by maximising the likelihood function. In order to simplify the calculations a logarithmic transformation of the likelihood function is normally performed; i.e. the estimators are obtained by maximising

$$L(\theta) = \sum_{i=1}^{n} \ln\left[f(x_i;\theta)\right] \tag{5.10}$$

where $f(x)$ is the probability density function.

Maximum likelihood parameter estimators are asymptotically more efficient. However, small sample estimators may be less efficient and in some cases the maximum likelihood procedure becomes unstable. Often maximum likelihood estimators cannot be reduced to simple explicit formula, and hence numerical methods such as the Newton Raphson scheme must be applied. Expressions for calculation of the maximum likelihood estimators for the different distributions are given in Appendix A.

6 GOODNESS-OF-FIT STATISTICS

For evaluating the fit of different distributions applied to the extreme value series, EVA calculates five goodness-of-fit statistics

- Chi-squared test statistic
- Kolmogorov-Smirnov test statistic
- Standardised least squares criterion
- Probability plot correlation coefficient
- Log-likelihood measure

It must be emphasised that the choice of probability distribution should not rely solely on the goodness-of-fit. The fact that many distributions have similar form in their central parts but differ significantly in the tails emphasises that the goodness-of-fit is not sufficient. The choice of probability distribution is generally a compromise between contradictory requirements. Selection of a distribution with few parameters provides robust parameter estimates but the goodness-of-fit may not be satisfactory. On the other hand, when selecting a distribution with more parameters, the goodness-of-fit will generally improve but at the expense of a large sampling uncertainty of the parameter estimates.

Besides an evaluation of the goodness-of-fit statistics, a graphical comparison of the different distributions with the observed extreme value series should be carried out. In this respect the histogram/frequency plot and the probability plot are useful. These plots are described in Section 8.

6.1 Chi-squared test

The χ^2-test statistic is based on a comparison of the number of observed events and the number of expected events (according to the specified probability distribution) in class intervals covering the range of the variable. The test statistic reads

$$z = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i} \tag{6.1}$$

where k is the number of classes, n_i is the number of observed events in class i, n is the sample size, and p_i is the probability corresponding to class i, implying that the number of expected events in class i is equal to np_i. The test is more powerful if the range of the variable is divided into classes of equal probability, i.e. $p = 1/k$. The corresponding class limits for the considered distributions are obtained from the quantile function cf. (2.3). The number of classes is determined such that the expected number of events in a class is not smaller than 5.

The test statistic is approximately χ^2-distributed with $k–1–q$ degrees of freedom where q is the number of estimated parameters. Thus, the H_0 hypothesis that data are distributed according to the specified probability distribution is rejected at significance level α if $z > \chi^2(k–1–q)_{1-\alpha}$ where $\chi^2(k–1–q)_{1-\alpha}$ is the $(1-\alpha)$-quantile in the χ^2-distribution with $k–1–q$ degrees of freedom.

6.2 Kolmogorov-Smirnov test

The Kolmogorov-Smirnov test is based on the deviation between the empirical and the theoretical distribution function. The test statistic is given by

$$z = Max\left|F_n(x) - F(x)\right| \tag{6.2}$$

where $F(x)$ is the theoretical cumulative distribution function, and $F_n(x)$ is the empirical distribution function defined as

$$F_n(x) = \begin{cases} 0 & , x < x_{(1)} \\ \dfrac{i}{n} & , x_{(i)} \le x < x_{(i+1)} \\ 1 & , x \ge x_{(n)} \end{cases} \tag{6.3}$$

For known distribution parameters, the distribution of the Kolmogorov-Smirnov statistic is independent of the considered distribution, and general tables of critical values of the test statistic can be used for evaluation of the significance level. In Table 6.1 critical values are given for the modified form of the test statistic [Stephens, 1986]

$$z^* = z\left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}\right) \tag{6.4}$$

When the distribution parameters are unknown and have to be estimated from the data, the distribution of the test statistic depends on the considered distribution, the estimated parameters, the estimation method, and the sample size. In this case no general table of critical values of the test statistic exists. In EVA, critical values based on Table 6.1 are calculated. However, since the parameters of the considered distributions are estimated from the data, the outcome of the test should not be used as a strict significance test.

Table 6.1 Critical values of the modified Kolmogorov-Smirnov test statistic in (6.4) [Stephens, 1986].

Significance level	0.25	0.15	0.10	0.05	0.025	0.01	0.005	0.001
Critical value	1.019	1.138	1.224	1.358	1.480	1.628	1.731	1.950

6.3 Standardised least squares criterion

The standardised least squares criterion (SLSC) and the probability plot correlation coefficient described in Section 6.4 are both based on the difference between the ordered observations and the corresponding order statistics for the considered probability distribution. The SLSC is defined using a reduced variate u_i (Takasao et al., 1986)

$$u_i = g(x_i;\theta) \tag{6.5}$$

where $g(.)$ is the transformation function, and θ are the distribution parameters. Expressions of the reduced variate for the different distributions included in EVA are given in Appendix A.

For the ordered observations $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$, the reduced variates u_i are calculated from (6.5) using the estimated parameters. The corresponding order statistics are given by

$$u_i^* = g(F^{-1}(p_i)) \tag{6.6}$$

where p_i is the probability of the i'th largest observation in a sample of n variables. The probability is determined by using a plotting position formula (see Section 8).

The SLSC is calculated as

$$z = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(u_{(i)} - u_i^*)^2}}{\left|u_{1-p}^* - u_p^*\right|} \tag{6.7}$$

where $u^*_{1\text{-}p}$ and u^*_p are the reduced variates calculated from (6.6) using non-exceedance probabilities $1–p$ and p, respectively. The denominator in (6.7) is introduced in order to standardise the measure, so that the SLSC can be used to compare goodness-of-fit between different distributions. Smaller values of SLSC correspond to better fits. In EVA, $p = 0.01$ is used for calculation of SLSC.

Formulae of the reduced variates and corresponding order statistics for the distributions available in EVA are given in Appendix A. For some distributions several formulations of the reduced variate have been proposed. In EVA, the SLSC1 formula is used as main output, whereas the other SLSC measures are given as supplementary output. It should be noted that for a consistent and more direct comparison between different distributions, the same reduced variate should be used, if possible. For instance, for comparing the goodness-of-fit between the Gumbel, Frechét, generalised extreme value, and square-root exponential distributions the SLSC measure based on the Gumbel reduced variate $u_i = –\ln[–\ln(p_i)]$ should be applied. For comparison of the exponential, generalised Pareto, and Weibull distributions the exponential reduced variate $u_i = –\ln(1–p_i)$ should be used.

The distribution of the SLSC statistic depends, in general, on the considered distribution, the estimated parameters, the estimation method, and the sample size. Thus, no general table for critical values of the test statistic exists.

In certain situations, some data points may fall outside the estimated range of the considered distributions (e.g. some observations are smaller (or larger) than the estimated location parameter), implying that the reduced variate is not defined. In EVA, these points are not included in the calculation of the SLSC measure. In such cases one should be careful in using the SLSC measure for comparing the goodness-of-fit of various distributions.

6.4 Probability plot correlation coefficient

The probability plot correlation coefficient (PPCC) [Vogel, 1986] is a measure of the correlation between the ordered observations $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$, and the corresponding order statistics

$$M_i = F^{-1}(p_i) \tag{6.8}$$

where p_i is the probability of the i'th largest observation in a sample of n variables. The probability is determined by using a plotting position formula (see Section 8). The PPCC is given by

$$z = \frac{\sum_{i=1}^{n}(x_{(i)} - \bar{x})(M_i - \overline{M})}{\left[\sum_{i=1}^{n}(x_{(i)} - \bar{x})^2 \sum_{i=1}^{n}(M_i - \overline{M})^2\right]^{1/2}} \tag{6.9}$$

where $\bar{x}$ and $\overline{M}$ are the sample mean values of the x_i and the M_i, respectively. Values of PPCC closer to unity correspond to better fits.

The distribution of the PPCC statistic depends, in general, on the considered distribution, the estimated parameters, the estimation method, and the sample size, and hence no general table for critical values of the test statistic exists. For the log-normal, Gumbel and Pearson Type 3 distributions, the distribution of the test statistic has been evaluated [Vogel, 1986; Vogel and McMartin, 1991].

Another formulation of the PPCC measure is based on the reduced variate defined above [Takara and Stedinger, 1994]. In this case the PPCC is given by

$$z = \frac{\sum_{i=1}^{n}(u_{(i)} - \bar{u})(u_i^* - \bar{u}^*)}{\left[\sum_{i=1}^{n}(u_{(i)} - \bar{u})^2 \sum_{i=1}^{n}(u_i^* - \bar{u}^*)^2\right]^{1/2}} \tag{6.10}$$

where $u_{(i)}$ and u_i* are the ordered reduced variate and the corresponding order statistic defined in (6.5)-(6.6). If the reduced variate is a linear transformation of the variable X, the two PPCC measures in (6.9) and (6.10) are identical.

As for the SLSC measure, in certain situations some data points may fall outside the estimated range of the considered distributions, implying that the reduced variate used in (6.10) is not defined. In EVA, these points are not included in the calculation of the PPCC measure.

6.5 Log-likelihood measure

The log-likelihood measure is given by

$$z = \sum_{i=1}^{n} \ln\left[f(x_i; \hat{\theta}) \right] \tag{6.11}$$

where $f(.)$ is the probability density function of the considered distribution, and $\hat{\theta}$ are the estimated parameters. Larger values of the log-likelihood measure correspond to better fits.

As noted above, in some cases data points may fall outside the estimated range of the probability distribution. For such points the probability density function equals zero, implying that (6.11) cannot be evaluated properly. In EVA, a corrected log-likelihood measure is calculated

$$z^* = \frac{n}{n-k} \sum_{i=1}^{n-k} \ln\left[f(x_i; \hat{\theta}) \right] \tag{6.12}$$

where k is the number of data points for which $f(x) = 0$, and the summation is performed for the $n{-}k$ data points where $f(x) > 0$.

7 UNCERTAINTY CALCULATIONS

Two different methods are available in EVA for evaluating the uncertainty of quantile estimates

- Monte Carlo simulation
- Jackknife resampling

7.1 Monte Carlo simulation

In Monte Carlo simulation the bias and the standard deviation of the quantile estimate is obtained by randomly generating a large number of samples that has the same statistical characteristics as the observed sample. The algorithm can be summarised as follows:

1 Randomly generate a set of *m* data points from the considered distribution using the estimated parameters, i.e.

$$x_i = F^{-1}(r_i;\hat{\theta}) \quad , \quad i = 1,2,..,m \tag{7.1}$$

where r_i is a randomly generated number between 0 and 1.

In the case of AMS or PDS with a fixed number of events, *m* is set equal to the sample size $m = n$. In the case of PDS with a fixed threshold level, the number of events is a random variable that is assumed to be Poisson distributed. In this case *m* is randomly generated from a Poisson distribution with parameter $\hat{\lambda} t$ where $\hat{\lambda}$ is the estimated average annual number of events for the observed sample, and *t* is the observation period. The average annual number of events for the generated sample (denoted sample no. *j*) is estimated as

$$\hat{\lambda}^{(j)} = \frac{m}{t} \tag{7.2}$$

2 From the generated sample, the parameters of the distribution are estimated. In the case of AMS, the *T*-year event estimate is then obtained from (2.5)

$$\hat{x}_T^{(j)} = F^{-1}\left(1 - \frac{1}{T};\hat{\theta}^{(j)}\right) \tag{7.3}$$

where $\hat{\theta}^{(j)}$ are the estimated parameters. In the case of PDS with a fixed threshold level, the *T*-year event estimate is obtained from (2.8)

$$\hat{x}_T^{(j)} = x_0 + F^{-1}\left(1 - \frac{1}{\hat{\lambda}^{(j)} T}; \hat{\theta}^{(j)}\right) \tag{7.4}$$

For PDS with a fixed number of events, the T-year event estimate is obtained from (2.9)

$$\hat{x}_T^{(j)} = F^{-1}\left(1 - \frac{1}{\lambda T}; \hat{\theta}^{(j)}\right) \tag{7.5}$$

3 Steps (1)-(2) are repeated *k* times. The mean and the standard deviation s_T of the *T*-year event estimate are then given by

$$\begin{aligned} \tilde{x}_T &= \bar{x}_T = \frac{1}{k}\sum_{j=1}^{k} \hat{x}_T^{(j)} \\ s_T^2 &= \frac{1}{k}\sum_{j=1}^{k} \left(\hat{x}_T^{(j)} - \bar{x}_T\right)^2 \end{aligned} \tag{7.6}$$

Investigations suggest that the Monte Carlo based estimates of the mean and the standard deviation of the *T*-year event estimator saturate at a sample size in the order of 10,000. Thus, in EVA the number of generated samples is set equal to k = 10,000.

In some cases, samples may be generated from which distribution parameters cannot be estimated, e.g. due to the generation of sample moments for which the distribution is not defined or due to the non-existence of an optimum of the likelihood function. Non-convergence of the optimisation algorithm is a common problem for the maximum likelihood procedure and is especially pronounced for small sample sizes [Madsen et al., 1997]. Another problem related to the Monte Carlo method is the generation of unreasonable *T*-year events, resulting in unreliable estimates of the mean and the standard deviation of the *T*-year event estimator. To circumvent this problem, samples that result in *T*-year event estimates larger than the event corresponding to a return period of 10,000 times *T* are excluded.

7.2 Jackknife resampling

In the jackknife resampling method the bias and the standard deviation of the quantile estimate is calculated by sampling n data sets of $(n-1)$ elements from the original data set. The algorithm can be summarised as follows:

1 From the original sample data element no. j is excluded.

2 The distribution parameters $\hat{\theta}^{(j)}$ are estimated from the sample $\{x_1, x_2, .., x_{j-1}, x_{j+1}, .., x_n\}$. In the case of AMS, the T-year event estimate is then obtained from (2.5)

$$\hat{x}_T^{(j)} = F^{-1}\left(1-\frac{1}{T};\hat{\theta}^{(j)}\right) \tag{7.7}$$

In the case of PDS with a fixed threshold level, the T-year event estimate is obtained from (2.8)

$$\hat{x}_T^{(j)} = x_0 + F^{-1}\left(1-\frac{1}{\hat{\lambda}T};\hat{\theta}^{(j)}\right) \tag{7.8}$$

Note that with this method it is not possible to include the uncertainty in the estimated number of extreme events. For PDS with a fixed number of events, the T-year event estimate is obtained from(2.9)

$$\hat{x}_T^{(j)} = F^{-1}\left(1-\frac{1}{\lambda T};\hat{\theta}^{(j)}\right) \tag{7.9}$$

3 Steps (1)-(2) are repeated n times ($j = 1,2,\ldots,n$). The jackknife estimate of the T-year event corrected for bias reads

$$\tilde{x}_T = n\hat{x}_T - (n-1)\bar{x}_T \quad , \quad \bar{x}_T = \frac{1}{n}\sum_{j=1}^{n}\hat{x}_T^{(j)} \tag{7.10}$$

where $\hat{x}_T$ is the T-year event estimate obtained from the original sample. The standard deviation s_T of the jackknife T-year event estimate is given by

$$s_T^2 = \frac{n-1}{n}\sum_{j=1}^{n}\left(\hat{x}_T^{(j)} - \bar{x}_T\right)^2 \tag{7.11}$$

8 FREQUENCY AND PROBABILITY PLOTS

8.1 Plot of histogram and probability density function

A histogram is a plot of the empirical probability density function. The histogram is constructed by dividing the range of the variable in class intervals and counting the number of observations in each class. Denoting by n_i the number of observations in class i, and Δx the size of the interval, the histogram value of class i is given by

$$f_i = \frac{n_i}{n\Delta x} \tag{8.1}$$

where n is the total number of observations. The appropriate number of classes k is determined from the following rule of thumb

$$k = \text{int}\left(1 + 3.3\log_{10}(n)\right) \tag{8.2}$$

where int(.) denotes nearest integer value.

For evaluating the goodness-of-fit of an estimated probability distribution, the probability density function is compared to the histogram.

8.2 Probability plots

A probability plot is a plot of the ordered observations $\{x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}\}$ versus an approximation of their expected values $F^{-1}(p_i)$, where p_i is the probability of the i'th largest observation in a sample of n variables. The probability is determined by using a plotting position formula.

The plotting position formulae available in EVA are shown in Table 8.1. These formulae can be written in a general form

$$p_i = \frac{i-a}{n+1-2a} \tag{8.3}$$

Table 8.1 Plotting position formulae.

Name	Formula	a
Weibull	$p_i = \dfrac{i}{n+1}$	0
Hazen	$p_i = \dfrac{i-0.5}{n}$	0.5
Gringorten	$p_i = \dfrac{i-0.44}{n+0.12}$	0.44
Blom	$p_i = \dfrac{i-0.375}{n+0.25}$	0.375
Cunnane	$p_i = \dfrac{i-0.40}{n+0.20}$	0.40

For plotting, three different probability papers are available: Gumbel, log-normal, and semi-log papers. In the Gumbel probability paper, the observations are plotted versus the Gumbel reduced variate

$$u_i^* = -\ln[-\ln(p_i)] \tag{8.4}$$

In the log-normal probability paper, the logarithmic transformed observations are plotted versus the standard normal variate

$$u_i^* = \Phi^{-1}(p_i) \tag{8.5}$$

In the semi-log probability paper, the observations are plotted versus the exponential reduced variate

$$u_i^* = -\ln(1-p_i) \tag{8.6}$$

Probability plots are used for evaluating the goodness-of-fit of the estimated probability distributions. In a Gumbel probability paper, the Gumbel distribution is a straight line, whereas the 2-parameter log-normal and the exponential distributions are straight lines in the log-normal and semi-log probability papers, respectively. For the other distributions available in

EVA, no general probability papers exist, since the shape of these distributions is variable. When plotted in one of the available probability papers, distributions with a variable shape are curved lines.

When evaluating the goodness-of-fit in a probability plot, also confidence levels of the considered distribution can be shown. The T-year event estimate is asymptotically normally distributed with mean $\tilde{x}_T$ and standard deviation s_T which are quantified using Monte Carlo simulation, cf. (7.6) or jackknife resampling, cf. (7.10)-(7.11). Approximate $(1-\alpha)$-confidence levels are then given by

$$\tilde{x}_T \pm q s_T \quad , \quad q = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \tag{8.7}$$

For instance, approximate 68% and 95% confidence levels correspond to $q = 1$ and $q = 2$, respectively.

9 REFERENCES

/1/ Bernardo, J.M., 1976, Algorithm AS 103: psi (digamma) function, *Appl. Statist., 25*, 315-317.

/2/ Bobée, B., 1975, The log Pearson Type 3 distribution and its application in hydrology, *Water Resour. Res., 11*(5), 681-689.

/3/ Bobée, B. and Robitaille, R., 1975, Correction of bias in the estimation of the coefficient of skewness, *Water Resour. Res., 11*(6), 851-854.

/4/ Etoh, T., Murota, A. and Nakanishi, M., 1987, SQRT-exponential type distribution of maximum, In: *Hydrologic Frequency Modeling* (ed V.P. Singh), D. Reidel Pub. Co., 253-264.

/5/ Gumbel, E.J., 1954, Statistical theory of droughts, *Hydraulics Division, ASCE*, *439*(HY), 1-19.

/6/ Hart, J.F. et al., 1968, *Computer Approximations*, Wiley, New York.

/7/ Hosking, J.R.M, 1985, Algorithm AS215: Maximum-likelihood estimation of the parameters of the generalized extreme-value distribution, *Applied Statist., 34*, 301-310.

/8/ Hosking, J.R.M., 1990, L-moments: Analysis and estimation of distributions using linear combinations of order statistics, *J. Royal Statist. Soc. B, 52*(1), 105-124.

/9/ Hosking, J.R.M., 1991, Fortran routines for use with the method of L-moments, *Res. Report RC17097*, IBM Research Division, Yorktown Heights, New York.

/10/ Hosking, J.R.M. and Wallis, J.R., 1987, Parameter and quantile estimation for the generalized Pareto distribution, *Technometrics, 29*(3), 339-349.

/11/ Hosking, J.R.M. and Wallis, J.R., 1997, *Regional Frequency Analysis, An Approach Based on L-Moments*, Cambridge University Press.

/12/ Hosking, J.R.M., Wallis, J.R. and Wood, E.F., 1985, Estimation of the generalized extreme-value distribution by the method of probability-weighted moments, *Technometrics, 27*(3), 251-261.

/13/ Ishihara, T. and Takase, N., 1957, The logarithmic normal distribution and its solution based on moment method, *Trans. JSCE, 47*, 18-23 (In Japanese).

/14/ Iwai, S., 1947, On the asymmetric distribution in hydrology, Collection of Treaties, *J. Civil Eng. Soc., 2*, 93-116 (In Japanese).

/15/ Kadoya, 1962, On the applicable ranges and parameters of logarithmic normal distributions of the Slade type, *Nougyou Doboku Kenkyuu, Extra Publication, 3*, 12-27 (In Japanese).

/16/ Landwehr, J.M, Matalas, N.C. and Wallis, J.R., 1979, Probability weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles, *Water Resour. Res., 15*(5), 1055-1064.

/17/ Madsen, H., Rasmussen, P.F. and Rosbjerg, D., 1997, Comparison of annual maximum series and partial duration series methods for modeling extreme hydrologic events. 1. At-site modeling, *Water Resour. Res., 33*(4), 747-757.

/18/ Pike, M.C. and Hill, I.D., 1966, Algorithm 291: logarithm of the gamma function, *Commun. Assoc. Comput. Mach., 9*, 684.

/19/ Shea, B.L., 1988, Algorithm AS 239: chi-squared and incomplete gamma integral, *Appl. Statist., 37*, 466-473.

/20/ Stedinger, J.R., 1980, Fitting log normal distributions to hydrologic data, *Water Resour. Res., 16*(3), 481-490.

/21/ Stephens, M.A., 1986, Tests based on EDF statistics, In: *Goodness-of-fit Techniques* (eds. R.B. D'Agostino and M.A. Stephens), Marcel Dekker Inc., 97-193.

/22/ Takara, K.T. and Stedinger, J.R., 1994, Recent Japanese contributions to frequency analysis and quantile lower bound estimators, In: *Stochastic and Statistical Methods in Hydrology and Environmental Engineering* (ed. K.W. Hipel), Kluwer, Vol .1, 217-234

/23/ Takasao, T., Takara, K. and Shimizu, A., 1986, A basic study on frequency analysis of hydrologic data in the Lake Biwa basin, *Annuals, Disas. Prev. Res. Inst.*, Kyoto University, 29B-2, 157-171 (In Japanese).

/24/ Vogel, R.M., 1986, The probability plot correlation coefficient test for the normal. lognormal, and Gumbel distributional hypotheses, *Water Resour. Res.*, 22(4), 587-590. Correction, *Water Resour. Res.*, 23(10), 2013.

/25/ Vogel, R.M. and McMartin, D.E., 1991, Probability plot goodness-of-fit and skewness estimation procedures for the Pearson Type 3 distribution, *Water Resour. Res.*, 27(12), 3149-3158.

/26/ Wichura, M., 1988, Algorithm AS 241: the percentage points of the normal distribution, *Appl. Statist., 37*, 477-484.

APPENDIX A

Probability distributions

For each of the distributions available in EVA the following is provided in this appendix

- Probability density function $f(x)$
- Cumulative distribution function $F(x)$
- Quantile function x_p corresponding to the non-exceedance probability p
- Expressions of ordinary moments and L-moments
- Description of parameter estimation by the method of moments, the method of L-moments and the maximum likelihood method
- Reduced variate u_p for calculation of standardised least squares (SLSC) goodness-of-fit criterion

In addition, the appendix includes descriptions of the different auxiliary functions used in EVA

- Gamma function
- Euler's psi function
- Incomplete gamma integral
- Cumulative distribution function of the standard normal distribution
- Quantile function of the standard normal distribution

A.1 EXPONENTIAL DISTRIBUTION

Definition

Parameters: ξ (location), α (scale)

Range: $\alpha > 0$, $\xi \leq x < \infty$

$$f(x) = \frac{1}{\alpha}\exp\left[-\frac{x-\xi}{\alpha}\right] \tag{A.1.1}$$

$$F(x) = 1-\exp\left[-\frac{x-\xi}{\alpha}\right] \tag{A.1.2}$$

$$x_p = \xi - \alpha\ln(1-p) \tag{A.1.3}$$

Moments

$$\mu = \xi + \alpha \tag{A.1.4}$$

$$\sigma^2 = \alpha^2 \tag{A.1.5}$$

L-moments

$$\lambda_1 = \xi + \alpha \tag{A.1.6}$$

$$\lambda_2 = \frac{\alpha}{2} \tag{A.1.7}$$

Moment estimates

If ξ is known, α is estimated from the sample mean value

$$\hat{\alpha} = \hat{\mu} - \xi \tag{A.1.8}$$

If ξ is unknown, moment estimates are given by

$$\hat{\alpha} = \hat{\sigma} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha} \tag{A.1.9}$$

L-moment estimates

If ξ is known, the L-moment estimate of α is identical to the moment estimate. If ξ is unknown, L-moment estimates are given by

$$\hat{\alpha} = 2\hat{\lambda}_2 \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \hat{\alpha} \tag{A.1.10}$$

Maximum likelihood estimates

If ξ is known, the maximum likelihood estimate of α is identical to the moment and the L-moment estimate.

Reduced variate

$$\text{SLSC1}: \qquad u_p = \frac{x_p - \xi}{\alpha} = -\ln(1-p) \tag{A.1.11}$$

A.2 GENERALISED PARETO DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\alpha > 0$, $\xi \le x < \infty$ for $\kappa < 0$, $\xi \le x \le \xi+\alpha/\kappa$ for $\kappa > 0$

Special case: Exponential distribution for $\kappa = 0$

$$f(x) = \frac{1}{\alpha}\left[1 - \kappa\frac{x-\xi}{\alpha}\right]^{1/\kappa - 1} \tag{A.2.1}$$

$$F(x) = 1 - \left[1 - \kappa\frac{x-\xi}{\alpha}\right]^{1/\kappa} \tag{A.2.2}$$

$$x_p = \xi + \frac{\alpha}{\kappa}\left[1-(1-p)^{\kappa}\right] \tag{A.2.3}$$

Moments

$$\mu = \xi + \frac{\alpha}{1+\kappa} \tag{A.2.4}$$

$$\sigma^2 = \frac{\alpha^2}{(1+\kappa)^2(1+2\kappa)} \tag{A.2.5}$$

$$\gamma_3 = \frac{2(1-\kappa)\sqrt{1+2\kappa}}{(1+3\kappa)} \tag{A.2.6}$$

L-moments

$$\lambda_1 = \xi + \frac{\alpha}{1+\kappa} \tag{A.2.7}$$

$$\lambda_2 = \frac{\alpha}{(1+\kappa)(2+\kappa)} \tag{A.2.8}$$

$$\tau_3 = \frac{(1-\kappa)}{(3+\kappa)} \tag{A.2.9}$$

Moment estimates

If ξ is known, moment estimates of α and κ are given by

$$\hat{\kappa} = \frac{1}{2}\left[\left(\frac{\hat{\mu}-\xi}{\hat{\sigma}}\right)^2 - 1\right] \quad , \quad \hat{\alpha} = (\hat{\mu}-\xi)(1+\hat{\kappa}) \tag{A.2.10}$$

If ξ is unknown, κ is estimated from the skewness estimator cf. (A.2.6) using a Newton-Raphson iteration scheme. Moment estimates of ξ and α are subsequently obtained from

$$\hat{\alpha} = \hat{\sigma}(1+\hat{\kappa})\sqrt{1+2\hat{\kappa}} \quad , \quad \hat{\xi} = \hat{\mu} - \frac{\hat{\alpha}}{1+\hat{\kappa}} \tag{A.2.11}$$

L-moment estimates

If ξ is known, L-moment estimates of α and κ are given by

$$\hat{\kappa} = \frac{\hat{\lambda}_1 - \xi}{\hat{\lambda}_2} - 2 \quad , \quad \hat{\alpha} = (\hat{\lambda}_1 - \xi)(1+\hat{\kappa}) \tag{A.2.12}$$

If ξ is unknown, L-moment estimates are given by

$$\hat{\kappa} = \frac{1-3\hat{\tau}_3}{1+\hat{\tau}_3} \quad , \quad \hat{\alpha} = \hat{\lambda}_2(1+\hat{\kappa})(2+\hat{\kappa}) \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \frac{\hat{\alpha}}{1+\hat{\kappa}} \tag{A.2.13}$$

Maximum likelihood estimates

The log-likelihood function reads

$$L = -n \ln \alpha + \frac{1-\kappa}{\kappa} \sum_{i=1}^{n} \ln\left[1 - \frac{\kappa}{\alpha}(x_i - \xi)\right] \tag{A.2.14}$$

If ξ is known, the maximum likelihood estimates are obtained by solving

$$\frac{\partial L}{\partial \alpha} = 0 \quad , \quad \frac{\partial L}{\partial \kappa} = 0 \tag{A.2.15}$$

using a modified Newton-Raphson iteration scheme [Hosking and Wallis, 1987].

Reduced variate

$$\text{SLSC1:} \qquad u_p = -\frac{1}{\kappa} \ln\left[1 - \kappa \frac{x_p - \xi}{\alpha}\right] = -\ln(1-p) \tag{A.2.16}$$

$$\text{SLSC2:} \qquad u_p = \kappa \frac{x_p - \xi}{\alpha} = 1 - (1-p)^{\kappa} \tag{A.2.17}$$

$$\text{SLSC3:} \qquad u_p = \left[1 - \kappa \frac{x_p - \xi}{\alpha}\right]^{1/\kappa} = 1 - p \tag{A.2.18}$$

A.3 GUMBEL DISTRIBUTION

Definition

Parameters: ξ (location), α (scale)

Range: $\alpha > 0$, $-\infty < x < \infty$

$$f(x) = \frac{1}{\alpha}\exp\left[-\frac{x-\xi}{\alpha} - \exp\left(-\frac{x-\xi}{\alpha}\right)\right] \tag{A.3.1}$$

$$F(x) = \exp\left[-\exp\left(-\frac{x-\xi}{\alpha}\right)\right] \tag{A.3.2}$$

$$x_p = \xi - \alpha \ln\left[-\ln(p)\right] \tag{A.3.3}$$

Moments

$$\mu = \xi + \alpha\gamma_E \tag{A.3.4}$$

$$\sigma^2 = \frac{\pi^2\alpha^2}{6} \tag{A.3.5}$$

where $\gamma_E = 0.5772\ldots$ is Euler's constant.

L-moments

$$\lambda_1 = \xi + \alpha\gamma_E \tag{A.3.6}$$

$$\lambda_2 = \alpha \ln 2 \tag{A.3.7}$$

Moment estimates

Moment estimates of ξ and α are obtained from (A.3.4)-(A.3.5)

$$\hat{\alpha} = \frac{\sqrt{6}\hat{\sigma}}{\pi} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha}\gamma_E \tag{A.3.8}$$

Gumbel (1954) proposed a least squares estimation method based on the linear relationship between the ordered observations and the corresponding order statistics based on the Gumbel reduced variate. This method can also be interpreted as a finite sample size correction to the moment estimates. The estimates of ξ and α are given by

$$\hat{\alpha} = \frac{\hat{\sigma}}{s_n} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha} m_n \tag{A.3.9}$$

where m_n and s_n are, respectively, the mean and the standard deviation of the order statistics based on the Gumbel reduced variate using the Weibull plotting position

$$u_i^* = -\ln\left[-\ln\left(\frac{i}{n+1} \right) \right] \quad , i = 1,2,..,n \tag{A.3.10}$$

For $n \to \infty$ the estimates in (A.3.9) converges to the moment estimates in (A.3.8).

L-moment estimates

L-moment estimates of ξ and α are obtained from (A.3.6)-(A.3.7)

$$\hat{\alpha} = \frac{\hat{\lambda}_2}{\ln 2} \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \hat{\alpha}\gamma_E \tag{A.3.11}$$

Maximum likelihood estimates

The maximum likelihood estimate of α is obtained by solving

$$\frac{\sum_{i=1}^{n} x_i \exp\left(-\frac{x_i}{\alpha} \right)}{\sum_{i=1}^{n} \exp\left(-\frac{x_i}{\alpha} \right)} + \alpha = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{A.3.12}$$

using Newton-Raphson iteration. The estimate of ξ is subsequently obtained from

$$\exp\left(\frac{\xi}{\alpha} \right) \sum_{i=1}^{n} \exp\left(-\frac{x_i}{\alpha} \right) = n \tag{A.3.13}$$

Reduced variate

$$\text{SLSC1}: \qquad u_p = \frac{x_p - \xi}{\alpha} = -\ln[-\ln(p)] \tag{A.3.14}$$

Truncated Gumbel Distribution

A truncated Gumbel distribution for modelling exceedances above the threshold level in the PDS can be defined by truncating the Gumbel distribution at the threshold level. The probability density function $g(x)$, cumulative distribution function $G(x)$ and the quantile function x_p are

$$g(x) = \frac{f(x)}{1 - F(x_0)} \tag{A.3.15}$$

$$G(x) = \frac{F(x) - F(x_0)}{1 - F(x_0)} \tag{A.3.16}$$

$$x_p = \xi - \alpha \ln\left[-\ln(F(x_0) - (1 - F(x_0))p)\right] \tag{A.3.17}$$

where x_0 is the threshold level, and $f(x)$ and $F(x)$ are the probability density function and cumulative distribution function, respectively, of the Gumbel distribution.

The maximum likelihood estimates of ξ and α are obtained by solving the following equations using Newton-Raphson iteration:

$$n\frac{F(x_0)\ln(F(x_0))}{1 - F(x_0)} + n + \sum_{i=1}^{n} \ln(F(x_i)) = 0 \tag{A.3.18}$$

$$\xi = \alpha \ln \left[\frac{n\left(\frac{1}{n}\sum_{i=1}^{n} x_i - x_0 - \alpha\right)}{\sum_{i=1}^{n} (x_i - x_0)\exp\left(-\frac{x_i}{\alpha}\right)} \right] \tag{A.3.19}$$

A.4 GENERALISED EXTREME VALUE DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\alpha > 0$, $\xi+\alpha/\kappa \le x < \infty$ for $\kappa < 0$, $-\infty \le x \le \xi+\alpha/\kappa$ for $\kappa > 0$

Special case: Gumbel distribution for $\kappa = 0$

$$f(x) = \frac{1}{\alpha}\left[1 - \frac{\kappa(x-\xi)}{\alpha}\right]^{1/\kappa - 1} \exp\left(-\left[1 - \frac{\kappa(x-\xi)}{\alpha}\right]^{1/\kappa}\right) \qquad \text{(A.4.1)}$$

$$F(x) = \exp\left(-\left[1 - \frac{\kappa(x-\xi)}{\alpha}\right]^{1/\kappa}\right) \qquad \text{(A.4.2)}$$

$$x_p = \xi + \frac{\alpha}{\kappa}\left(1 - \left[-\ln(p)\right]^{\kappa}\right) \qquad \text{(A.4.3)}$$

Moments

$$\mu = \xi + \frac{\alpha}{\kappa}\left[1 - \Gamma(1+\kappa)\right] \qquad \text{(A.4.4)}$$

$$\sigma^2 = \left(\frac{\alpha}{\kappa}\right)^2 \left[\Gamma(1+2\kappa) - \left[\Gamma(1+\kappa)\right]^2\right] \qquad \text{(A.4.5)}$$

$$\gamma_3 = \operatorname{sgn}(\kappa)\frac{-\Gamma(1+3\kappa) + 3\Gamma(1+\kappa)\Gamma(1+2\kappa) - 2\left[\Gamma(1+\kappa)\right]^3}{\left[\Gamma(1+2\kappa) - \left[\Gamma(1+\kappa)\right]^2\right]^{3/2}} \qquad \text{(A.4.6)}$$

where sgn(κ) is plus or minus 1 depending on the sign of κ, and $\Gamma(.)$ is the gamma function.

L-moments

$$\lambda_1 = \xi + \frac{\alpha}{\kappa}\left[1 - \Gamma(1+\kappa)\right] \tag{A.4.7}$$

$$\lambda_2 = \frac{\alpha}{\kappa}\left(1 - 2^{-\kappa}\right)\Gamma(1+\kappa) \tag{A.4.8}$$

$$\tau_3 = \frac{2\left(1 - 3^{-\kappa}\right)}{1 - 2^{-\kappa}} - 3 \tag{A.4.9}$$

Moment estimates

The shape parameter κ is estimated from the skewness estimator cf. (A.4.6) using a Newton-Raphson iteration scheme. In this scheme, an analytic expression of the derivative of the gamma function based on Euler's psi function is used. Moment estimates of ξ and α are subsequently obtained from

$$\hat{\alpha} = \frac{\hat{\sigma}\left|\hat{\kappa}\right|}{\sqrt{\Gamma(1+2\hat{\kappa}) - \left[\Gamma(1+\hat{\kappa})\right]^2}} \quad , \quad \hat{\xi} = \hat{\mu} - \frac{\hat{\alpha}}{\hat{\kappa}}\left[1 - \Gamma(1+\hat{\kappa})\right] \tag{A.4.10}$$

L-moment estimates

For estimation of the shape parameter κ the approximation given by Hosking [1991] is used which is an extension of the approximation presented by Hosking et al. [1985]

$$\hat{\kappa} = 7.817740c + 2.930462c^2 + 13.641492c^3 + 17.206675c^4 \tag{A.4.11}$$

where

$$c = \frac{2}{3 + \hat{\tau}_3} - \frac{\ln 2}{\ln 3} \tag{A.4.12}$$

If $\tau_3 < -0.1$ or $\tau_3 > 0.5$, the approximation is less accurate and Newton-Raphson iteration is applied for further refinement. L-moment estimates of ξ and α are subsequently obtained from

$$\hat{\alpha} = \frac{\hat{\lambda}_2 \hat{\kappa}}{(1-2^{-\hat{\kappa}})\Gamma(1+\hat{\kappa})} \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \frac{\hat{\alpha}}{\hat{\kappa}}\left[1-\Gamma(1+\hat{\kappa})\right] \qquad \text{(A.4.13)}$$

Maximum likelihood estimates

Maximum likelihood estimates of the GEV parameters are obtained using the modified Newton-Raphson algorithm presented by Hosking [1985].

Reduced variate

$$\text{SLSC1}: \qquad u_p = -\frac{1}{\kappa}\ln\left[1-\kappa\frac{x_p-\xi}{\alpha}\right] = -\ln\left[-\ln(p)\right] \qquad \text{(A.4.14)}$$

$$\text{SLSC2}: \qquad u_p = \kappa\frac{x_p-\xi}{\alpha} = 1-\left[-\ln(p)\right]^{\kappa} \qquad \text{(A.4.15)}$$

$$\text{SLSC3}: \qquad u_p = \left[1-\kappa\frac{x_p-\xi}{\alpha}\right]^{1/\kappa} = -\ln(p) \qquad \text{(A.4.16)}$$

A.5 WEIBULL DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\alpha > 0$, $\kappa > 0$, $\xi < x < \infty$

Special case: Exponential distribution for $\kappa = 1$

$$f(x) = \frac{\kappa}{\alpha}\left(\frac{x-\xi}{\alpha}\right)^{\kappa-1} \exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{\kappa}\right] \tag{A.5.1}$$

$$F(x) = 1 - \exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{\kappa}\right] \tag{A.5.2}$$

$$x_p = \xi + \alpha\left[-\ln(1-p)\right]^{1/\kappa} \tag{A.5.3}$$

The Weibull distribution is a reverse generalised extreme value distribution with parameters

$$\xi_{GEV} = \xi_{WEI} - \alpha_{WEI} \quad , \quad \alpha_{GEV} = \frac{\alpha_{WEI}}{\kappa_{WEI}} \quad , \quad \kappa_{GEV} = \frac{1}{\kappa_{WEI}} \tag{A.5.4}$$

where subscripts *GEV* and *WEI* refer to generalised extreme value and Weibull distributions, respectively.

Moments

$$\mu = \xi + \alpha\Gamma\left(1+\frac{1}{\kappa}\right) \tag{A.5.5}$$

$$\sigma^2 = \alpha^2\left[\Gamma\left(1+\frac{2}{\kappa}\right) - \left[\Gamma\left(1+\frac{1}{\kappa}\right)\right]^2\right] \tag{A.5.6}$$

$$\gamma_3 = \frac{\Gamma\left(1+\frac{3}{\kappa}\right) - 3\Gamma\left(1+\frac{1}{\kappa}\right)\Gamma\left(1+\frac{2}{\kappa}\right) + 2\left[\Gamma\left(1+\frac{1}{\kappa}\right)\right]^3}{\left[\Gamma\left(1+\frac{2}{\kappa}\right) - \left[\Gamma\left(1+\frac{1}{\kappa}\right)\right]^2\right]^{3/2}} \tag{A.5.7}$$

where $\Gamma(.)$ is the gamma function.

L-moments

$$\lambda_1 = \xi + \alpha\Gamma\left(1+\frac{1}{\kappa}\right) \tag{A.5.8}$$

$$\lambda_2 = \alpha\left(1-2^{-1/\kappa}\right)\Gamma\left(1+\frac{1}{\kappa}\right) \tag{A.5.9}$$

$$\tau_3 = 3 - \frac{2\left(1-3^{-1/\kappa}\right)}{1-2^{-1/\kappa}} \tag{A.5.10}$$

Moment estimates

If ξ is known, the moment estimate of κ is obtained by combining (A.5.5) and (A.5.6)

$$\frac{\hat{\sigma}^2}{(\hat{\mu}-\xi)^2} = \frac{\Gamma\left(1+\frac{2}{\hat{\kappa}}\right)}{\left[\Gamma\left(1+\frac{1}{\hat{\kappa}}\right)\right]^2} - 1 \tag{A.5.11}$$

which is solved using Newton-Raphson iteration. In this scheme, an analytic expression of the derivative of the gamma function based on Euler's psi function is used. The moment estimate of α is then given by

$$\hat{\alpha} = \frac{\hat{\mu} - \xi}{\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right)} \tag{A.5.12}$$

If ξ is unknown, the moment estimate of κ is obtained from the skewness estimator cf. (A.5.7) using Newton-Raphson iteration. The iterative scheme is similar to the one applied for estimation of the shape parameter of the GEV distribution using $-\gamma_3$ and $\kappa_{GEV} = 1/\kappa$. The skewness estimator is corrected according to the bias correction formula given by Bobée and Robitaille [1975]

$$\hat{\gamma}_3^* = (1+\beta)\hat{\gamma}_3 \quad , \quad \beta = \left(0.01 + \frac{5.05}{n} + \frac{20.13}{n^2}\right) + \left(\frac{0.69}{n} + \frac{27.15}{n^2}\right)\hat{\gamma}_3^3 \tag{A.5.13}$$

which is valid for $0.25 \le \gamma_3 \le 5.0$ and $20 \le n \le 90$. The bias correction factor β is shown in Fig A.5.1. If γ_3 or n fall outside the ranges of the Bobée-Robitaille formula, the skewness is corrected using the following general bias correction

$$\hat{\gamma}_3^* = \frac{\sqrt{n(n-1)}}{n-2}\hat{\gamma}_3 \tag{A.5.14}$$

Moment estimates of ξ and α are given by

$$\hat{\alpha} = \frac{\hat{\sigma}}{\sqrt{\Gamma\left(1 + \frac{2}{\hat{\kappa}}\right) - \left[\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right)\right]^2}} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha}\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right) \tag{A.5.15}$$

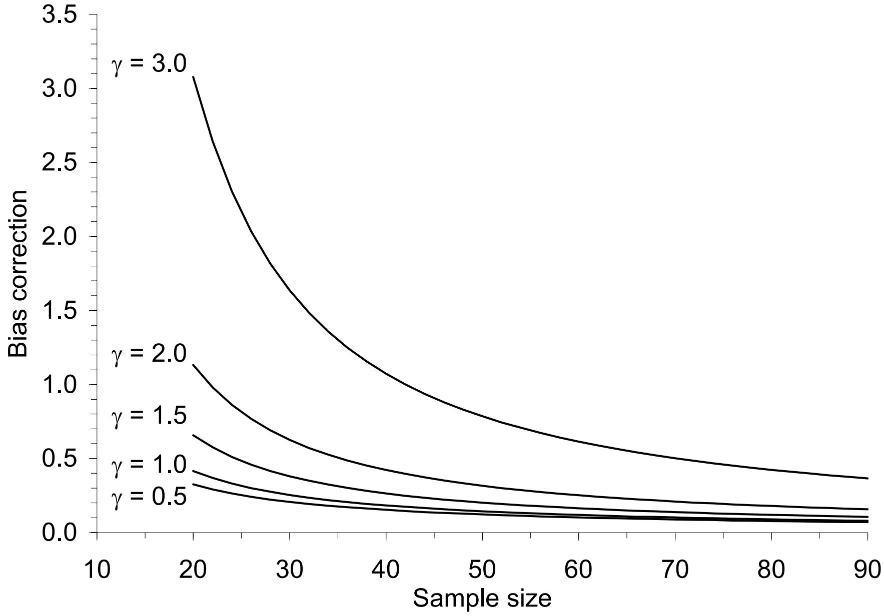

Fig A.5.1 *Bias correction factor β of the sample skewness γ for the Weibull distribution.*

L-moment estimates

If ξ is known, L-moment estimates of α and κ are given by

$$\hat{\kappa} = -\frac{\ln 2}{\ln\left(1 - \frac{\hat{\lambda}_2}{\hat{\lambda}_1}\right)} \quad , \quad \hat{\alpha} = \frac{\hat{\lambda}_1 - \xi}{\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right)} \tag{A.5.16}$$

If ξ is unknown, the shape parameter is estimated from the approximate formula (A.4.11) for estimation of the shape parameter of the GEV distribution using $-\tau_3$ and $\kappa_{GEV} = 1/\kappa$. L-moment estimates of ξ and α are then given by

$$\hat{\alpha} = \frac{\hat{\lambda}_2}{\left(1 - 2^{-1/\hat{\kappa}}\right)\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right)} \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \hat{\alpha}\Gamma\left(1 + \frac{1}{\hat{\kappa}}\right) \tag{A.5.17}$$

Maximum likelihood estimates

If ξ is known, the maximum likelihood estimate of κ is obtained by solving

$$\frac{1}{\kappa} = \frac{\sum_{i=1}^{n}(x_i - \xi)^{\kappa}\ln(x_i - \xi)}{\sum_{i=1}^{n}(x_i - \xi)^{\kappa}} - \frac{1}{n}\sum_{i=1}^{n}\ln(x_i - \xi) \tag{A.5.18}$$

using Newton-Raphson iteration. The maximum likelihood estimate of α is subsequently obtained from

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \xi)^{\hat{\kappa}}\right]^{1/\hat{\kappa}} \tag{A.5.19}$$

Reduced variate

$$\text{SLSC1}: \qquad u_p = \left(\frac{x_p - \xi}{\alpha}\right)^{\kappa} = -\ln(1-p) \tag{A.5.20}$$

$$\text{SLSC2}: \qquad u_p = \kappa \ln\left(\frac{x_p - \xi}{\alpha}\right) = \ln\left[-\ln(1-p)\right] \tag{A.5.21}$$

$$\text{SLSC3}: \qquad u_p = \frac{x_p - \xi}{\alpha} = \left[-\ln(1-p)\right]^{1/\kappa} \tag{A.5.22}$$

A.6 FRECHÉT DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\alpha > 0$, $\kappa > 0$, $\xi < x < \infty$

$$f(x) = \frac{\kappa}{\alpha}\left(\frac{x-\xi}{\alpha}\right)^{-(\kappa+1)} \exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right] \qquad \text{(A.6.1)}$$

$$F(x) = \exp\left[-\left(\frac{x-\xi}{\alpha}\right)^{-\kappa}\right] \qquad \text{(A.6.2)}$$

$$x_p = \xi + \alpha\left[-\ln(p)\right]^{-1/\kappa} \qquad \text{(A.6.3)}$$

Moments

$$\mu = \xi + \alpha\Gamma\left(1-\frac{1}{\kappa}\right) \qquad \text{(A.6.4)}$$

$$\sigma^2 = \alpha^2\left[\Gamma\left(1-\frac{2}{\kappa}\right) - \left[\Gamma\left(1-\frac{1}{\kappa}\right)\right]^2\right] \qquad \text{(A.6.5)}$$

$$\gamma_3 = \frac{\Gamma\left(1-\frac{3}{\kappa}\right) - 3\Gamma\left(1-\frac{1}{\kappa}\right)\Gamma\left(1-\frac{2}{\kappa}\right) + 2\left[\Gamma\left(1-\frac{1}{\kappa}\right)\right]^3}{\left[\Gamma\left(1-\frac{2}{\kappa}\right) - \left[\Gamma\left(1-\frac{1}{\kappa}\right)\right]^2\right]^{3/2}} \qquad \text{(A.6.6)}$$

where $\Gamma(.)$ is the gamma function. The Frechét distribution is defined only for skewness larger than the skewness of the Gumbel distribution, i.e. $\gamma_3 >$ 1.1396.

Moment estimates

For estimation of κ the method proposed by Kadoya [1962] is employed. A reduced variate y is defined as follows

$$y = \frac{x-\xi}{\alpha} = \exp\left(\frac{u}{\kappa}\right) \quad , \quad u = -\ln[-\ln(p)] \tag{A.6.7}$$

Since y is a linear transformation of x, the coefficient of skewness of y and x are identical. The expected value of the ordered sample $y_{(1)} \le y_{(2)} \le \ldots \le y_{(n)}$ is given by

$$E\{y_{(i)}\} = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} \Gamma\left(1-\frac{1}{\kappa}\right) \sum_{r=0}^{n-i} (-1)^r \frac{(n-i)!}{(n-i-r)!r!} (i+r)^{-1+1/\kappa} \tag{A.6.8}$$

An estimate of κ can now be found by solving

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\left(E\{y_{(i)}\} - \overline{E\{y_{(i)}\}}\right)^3}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(E\{y_{(i)}\} - \overline{E\{y_{(i)}\}}\right)^2\right]^{3/2}} = \hat{\gamma}_3 \quad , \quad \overline{E\{y_{(i)}\}} = \frac{1}{n}\sum_{i=1}^{n} E\{y_{(i)}\} \tag{A.6.9}$$

using iteration.

Since the computation of the expected value of y is numerically complicated, an approximation of the non-exceedance probability is introduced

$$F(E\{y_{(i)}\}) = F(E\{y_{(1)}\}) + \frac{i-1}{n-1}\left[F(E\{y_{(n)}\}) - F(E\{y_{(1)}\})\right] \tag{A.6.10}$$

where

$$E\{y_{(1)}\} = n\Gamma\left(1-\frac{1}{\kappa}\right)\left[1-(n-1)2^{-1+1/\kappa}+\frac{(n-1)(n-2)}{2}3^{-1+1/\kappa}-....\right]$$
$$E\{y_{(n)}\} = n\Gamma\left(1-\frac{1}{\kappa}\right)n^{-1+1/\kappa}$$

(A.6.11)

For sample sizes larger than about 40, numerical rounding errors become dominant for calculation of $E\{y_{(1)}\}$. Hence, for $n > 40$ an asymptotic approximation is used, assuming a symmetric non-exceedance probability

$$F(E\{y_{(1)}\}) = 1 - F(E\{y_{(n)}\}) \tag{A.6.12}$$

The approximated $E\{y_{(i)}\}$ to be used in (A.6.9) is finally obtained from (A.6.7)

$$E\{y_{(i)}\} = \exp\left(\frac{u_{(i)}}{\kappa}\right)$$
$$u_{(i)} = -\ln\left[-\ln\left(F(E\{y_{(1)}\}) + \frac{i-1}{n-1}\left[F(E\{y_{(n)}\}) - F(E\{y_{(1)}\})\right]\right)\right]$$

(A.6.13)

The estimation procedure can be interpreted as a bias correction to the skewness estimator. The bias correction factor β is given by

$$\hat{\gamma}_3^* = (1+\beta)\hat{\gamma}_3 \quad , \quad \beta = \frac{\hat{\gamma}_3^*}{\hat{\gamma}_3} - 1 \tag{A.6.14}$$

where $\hat{\gamma}_3^*$ is obtained from (A.6.6) using the estimated value of κ. The bias correction factor is shown in Fig A.6.2.

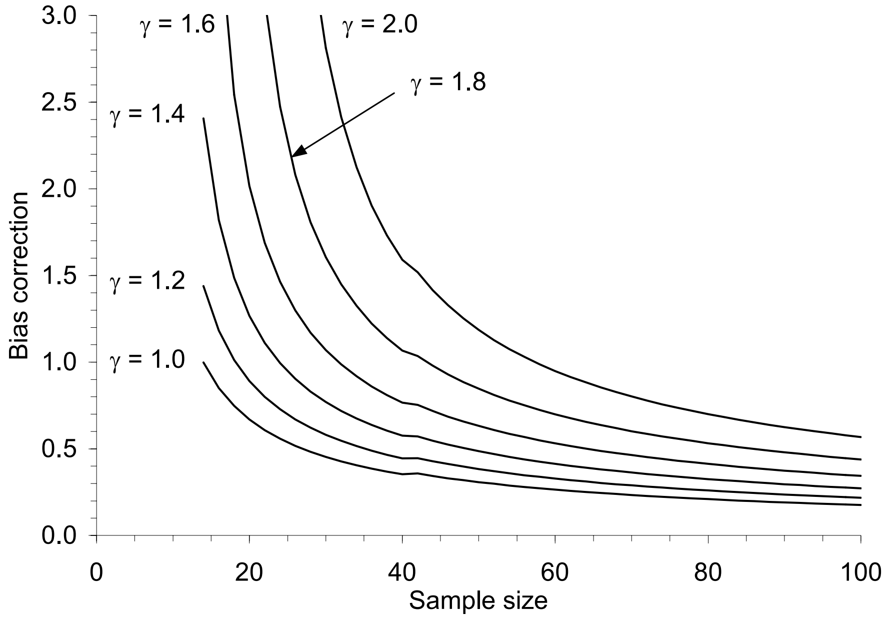

Fig A.6.2 Bias correction factor β of the sample skewness γ for the Frechét distribution.

Having estimated κ, moment estimates of ξ and α are subsequently obtained from

$$\hat{\alpha} = \frac{\hat{\sigma}}{\sqrt{\Gamma\left(1-\frac{2}{\hat{\kappa}}\right)-\left[\Gamma\left(1-\frac{1}{\hat{\kappa}}\right)\right]^2}} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha}\Gamma\left(1-\frac{1}{\hat{\kappa}}\right) \qquad \text{(A.6.15)}$$

Reduced variate

$$\text{SLSC1:} \qquad u_p = \kappa \ln\left(\frac{x_p - \xi}{\alpha}\right) = -\ln[-\ln(p)] \qquad \text{(A.6.16)}$$

A.7 GAMMA/PEARSON TYPE 3 DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\kappa > 0$, $\xi \le x < \infty$ for $\alpha > 0$, $-\infty \le x \le \xi$ for $\alpha < 0$

Special cases: Exponential distribution for $\kappa = 1$ and $\alpha > 0$. Normal distribution for $\gamma = 0$

$$f(x) = \frac{1}{|\alpha|\Gamma(\kappa)}\left(\frac{x-\xi}{\alpha}\right)^{\kappa-1}\exp\left(-\frac{x-\xi}{\alpha}\right) \tag{A.7.1}$$

$$F(x) = \begin{cases} G\left(\kappa, \dfrac{x-\xi}{\alpha}\right) & ,\alpha > 0 \\ 1 - G\left(\kappa, \dfrac{x-\xi}{\alpha}\right) & ,\alpha < 0 \end{cases} \tag{A.7.2}$$

$$x_p = \xi + \alpha u_p \tag{A.7.3}$$

where $\Gamma(.)$ is the gamma function, and $G(.,.)$ is the incomplete gamma integral. No explicit expression of the quantile function is available. The standardised quantile u_p is determined as the solution of $F(u) = p$ where $u = (x-\xi)/\alpha$ using Newton-Raphson iteration.

Moments

$$\mu = \xi + \alpha\kappa \tag{A.7.4}$$

$$\sigma^2 = \alpha^2\kappa \tag{A.7.5}$$

$$\gamma_3 = \begin{cases} \dfrac{2}{\sqrt{\kappa}} & ,\alpha > 0 \\ -\dfrac{2}{\sqrt{\kappa}} & ,\alpha < 0 \end{cases} \tag{A.7.6}$$

L-moments

$$\lambda_1 = \xi + \alpha\kappa \tag{A.7.7}$$

$$\lambda_2 = \frac{|\alpha|}{\sqrt{\pi}} \frac{\Gamma\left(\kappa + \frac{1}{2}\right)}{\Gamma(\kappa)} \tag{A.7.8}$$

$$\tau_3 = \begin{cases} 6I_{1/3}(\kappa, 2\kappa) - 3 & ,\alpha > 0 \\ -6I_{1/3}(\kappa, 2\kappa) + 3 & ,\alpha < 0 \end{cases} \tag{A.7.9}$$

where $I_x(.,.)$ is the incomplete beta function ratio. Rational-function approximations of τ_3 as a function of κ are given by Hosking and Wallis [1997].

Moment estimates

If ξ is known, moment estimates of α and κ are obtained from (A.7.4)-(A.7.5)

$$\hat{\kappa} = \frac{(\hat{\mu} - \xi)^2}{\hat{\sigma}^2} \quad , \quad \hat{\alpha} = \frac{\hat{\sigma}^2}{\hat{\mu} - \xi} \tag{A.7.10}$$

If ξ is unknown, the shape parameter κ is estimated from the skewness estimator cf. (A.7.6). The skewness estimator is corrected according to the bias correction formula given by Bobée and Robitaille [1975]

$$\hat{\gamma}_3^* = (1 + \beta)\hat{\gamma}_3 \quad , \quad \beta = \left(\frac{6.51}{n} + \frac{20.2}{n^2}\right) + \left(\frac{1.48}{n} + \frac{6.77}{n^2}\right)\hat{\gamma}_3^2 \tag{A.7.11}$$

which is valid for $0.25 \le \gamma_3 \le 5.0$ and $20 \le n \le 90$. The bias correction factor β is shown in Fig A.7.3. If γ_3 or n fall outside the ranges of the Bobée-Robitaille formula, the skewness is corrected using the following general bias correction

$$\hat{\gamma}_3^* = \frac{\sqrt{n(n-1)}}{n-2} \hat{\gamma}_3 \tag{A.7.12}$$

Moment estimates of ξ and α are obtained from (A.7.4)-(A.7.5)

$$\hat{\alpha} = \mathrm{sgn}(\hat{\gamma}_3^*)\frac{\hat{\sigma}}{\sqrt{\hat{\kappa}}} \quad , \quad \hat{\xi} = \hat{\mu} - \hat{\alpha}\hat{\kappa} \tag{A.7.13}$$

where sgn(.) is plus or minus 1, depending on the sign of $\hat{\gamma}_3^*$.

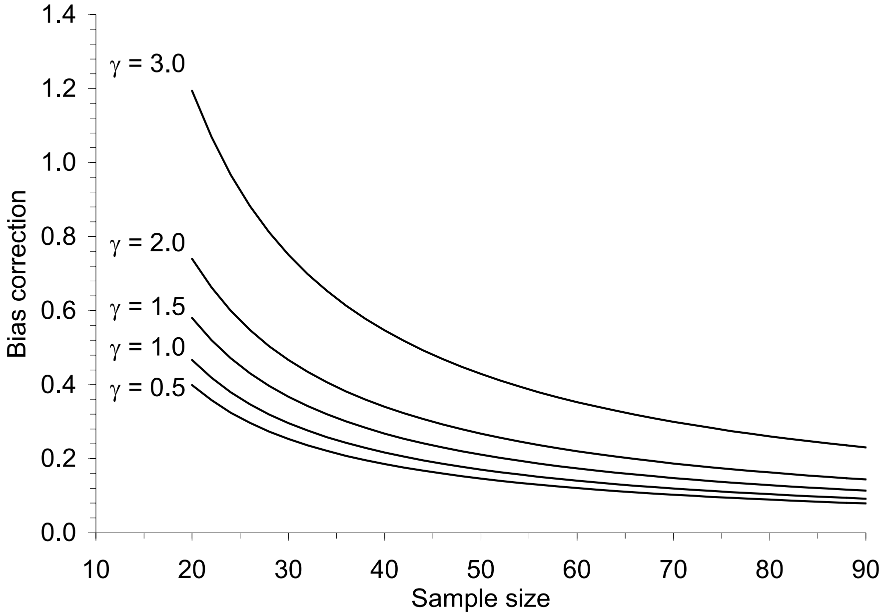

Fig A.7.3 Bias correction factor β of the sample skewness γ for the Pearson Type 3 distribution.

L-moment estimates

If ξ is known, L-moment estimates of α and κ are obtained from (A.7.7)-(A.7.8). For estimation of κ, rational-function approximations of κ as a function of the L-coefficient of variation τ_2 are applied [Hosking, 1991]

For $\tau_2 < ½$:

$$\kappa = \frac{1 + A_1 z}{z + A_2 z^2 + A_3 z^3} \quad , \quad z = \pi\tau_2^2 \quad , \quad \tau_2 = \frac{\lambda_2}{\lambda_1 - \xi} \tag{A.7.14}$$

For $\tau_2 \geq ½$:

$$\kappa = \frac{B_1 z + B_2 z^2}{1 + B_3 z + B_4 z^2} \quad , \quad z = 1 - \tau_2 \quad , \quad \tau_2 = \frac{\lambda_2}{\lambda_1 - \xi} \tag{A.7.15}$$

The coefficients of the rational functions are shown in Table A.7.1. The estimate of α is subsequently obtained from

$$\hat{\alpha} = \frac{\hat{\lambda}_1 - \xi}{\hat{\kappa}} \tag{A.7.16}$$

For estimation of κ when ξ is unknown, rational-function approximations of κ as a function of the L-skewness are applied [Hosking and Wallis, 1997]

For $|\tau_3| < 1/3$:

$$\kappa = \frac{1 + C_1 z}{z + C_2 z^2 + C_3 z^3} \quad , \quad z = 3\pi\tau_3^2 \tag{A.7.17}$$

For $|\tau_3| \geq 1/3$:

$$\kappa = \frac{D_1 z + D_2 z^2 + D_3 z^3}{1 + D_4 z + D_5 z^2 + D_6 z^3} \quad , \quad z = 1 - |\tau_3| \tag{A.7.18}$$

The coefficients of the rational functions are shown in Table A.7.1. The estimates of ξ and α are subsequently obtained from

$$\hat{\alpha} = \operatorname{sgn}(\hat{\tau}_3) \frac{\hat{\lambda}_2 \sqrt{\pi}\, \Gamma(\hat{\kappa})}{\Gamma\left(\hat{\kappa} + \frac{1}{2}\right)} \quad , \quad \hat{\xi} = \hat{\lambda}_1 - \hat{\alpha}\hat{\kappa} \tag{A.7.19}$$

where sgn(.) is plus or minus 1, depending on the sign of $\hat{\tau}_3$.

Table A.7.1 Coefficients of the rational-function approximations (A.7.14)-(A.7.15) and (A.7.17)-(A.7.18).

A_i	B_i	C_i	D_i
A_1=-0.3080	B_1=0.7213	C_1=0.2906	D_1=0.36067
A_2=-0.05812	B_2=-0.5947	C_2=0.1882	D_2=-0.59567
A_3=0.01765	B_3=-2.1817	C_3=0.0442	D_3=0.25361
	B_4=1.2113		D_4=-2.78861
			D_5=2.56096
			D_6=-0.77045

Maximum likelihood estimates

If ξ is known, maximum likelihood estimates are obtained from the following set of equations

$$\sum_{i=1}^{n}\ln(x_i-\xi)-n\ln\alpha-n\psi(\kappa)=0 \quad , \quad \alpha=\frac{1}{\kappa}\frac{1}{n}\sum_{i=1}^{n}(x_i-\xi) \qquad \text{(A.7.20)}$$

where $\psi(.)$ is Euler's psi function. An estimate of κ is found from the first equation using bisection.

Reduced variate

$$\text{SLSC1}: \qquad u_p=\frac{x_p-\xi}{\alpha} \quad , \quad p=\begin{cases}\dfrac{1}{\Gamma(\kappa)}G\left(\kappa,u_p\right) & ,\alpha>0\\ 1-\dfrac{1}{\Gamma(\kappa)}G\left(\kappa,u_p\right) & ,\alpha<0\end{cases}$$

(A.7.21)

A.8 LOG-PEARSON TYPE 3 DISTRIBUTION

Definition

Parameters: ξ (location), α (scale), κ (shape)

Range: $\kappa > 0$, $\exp(\xi) \le x < \infty$ for $\alpha > 0$, $0 \le x \le \exp(\xi)$ for $\alpha < 0$

Special case: 2-parameter log-normal distribution for $\gamma_y = 0$

If X is distributed according to a log-Pearson Type 3 distribution, then $Y = \ln(X)$ is Pearson Type 3 distributed. The parameters ξ, α and κ are, respectively, the location, scale and shape parameter of the corresponding Pearson Type 3 distribution.

$$f(x) = \frac{1}{x|\alpha|\Gamma(\kappa)}\left(\frac{\ln(x)-\xi}{\alpha}\right)^{\kappa-1}\exp\left(-\frac{\ln(x)-\xi}{\alpha}\right) \tag{A.8.1}$$

$$F(x) = \begin{cases} G\left(\kappa, \dfrac{\ln(x)-\xi}{\alpha}\right) & ,\alpha > 0 \\ 1-G\left(\kappa, \dfrac{\ln(x)-\xi}{\alpha}\right) & ,\alpha < 0 \end{cases} \tag{A.8.2}$$

$$x_p = \exp(\xi + \alpha u_p) \tag{A.8.3}$$

where $\Gamma(.)$ is the gamma function, and $G(.,.)$ is the incomplete gamma integral. No explicit expression of the quantile function is available. The standardised quantile u_p is determined as the solution of $F(u) = p$ where $u = (\ln(x)-\xi)/\alpha$ using Newton-Raphson iteration

Moment estimates

Moments in log-space

Parameter estimates are obtained from the sample moments of the logarithmic transformed data $\{y_i = \ln(x_i),\ i = 1,2,...,n\}$ using (A.7.11)-(A.7.13).

Moments in real space

Bobée [1975] proposed an estimation method based on the moments in real space. The moments about the origin are given by

$$\upsilon_r = \frac{\exp(r\xi)}{(1-r\alpha)^{\kappa}} \quad , \quad r = 1,2,3,... \tag{A.8.4}$$

The estimate of α is obtained from

$$\frac{3\ln(1-\hat{\alpha}) - \ln(1-3\hat{\alpha})}{2\ln(1-\hat{\alpha}) - \ln(1-2\hat{\alpha})} = \frac{\ln\hat{v}_3 - 3\ln\hat{v}_1}{\ln\hat{v}_2 - 2\ln\hat{v}_1} \tag{A.8.5}$$

where the sample moments are calculated as

$$\hat{v}_r = \frac{1}{n}\sum_{i=1}^{n} x_i^r \tag{A.8.6}$$

Eq. (A.8.5) is solved using a Newton-Raphson iteration scheme. Estimates of ξ and κ are subsequently obtained from

$$\hat{\kappa} = \frac{\ln\hat{v}_2 - 2\ln\hat{v}_1}{2\ln(1-\hat{\alpha}) - \ln(1-2\hat{\alpha})} \quad , \quad \hat{\xi} = \ln\hat{v}_1 + \hat{\kappa}\ln(1-\hat{\alpha}) \tag{A.8.7}$$

These estimates are corrected using a bias correction of the equivalent Pearson Type 3 skewness cf. (A.7.6) according to the Bobée and Robitaille [1975] formula.

L-moment estimates

Parameter estimates are obtained from the sample L-moments of the logarithmic transformed data $\{y_i = \ln(x_i), i = 1,2,...,n\}$ using (A.7.17)-(A.7.19).

Reduced variate

$$\text{SLSC1}: \quad u_p = \frac{\ln(x_p) - \xi}{\alpha} \quad , \quad p = \begin{cases} \frac{1}{\Gamma(\kappa)} G(\kappa, u_p) & ,\alpha > 0 \\ 1 - \frac{1}{\Gamma(\kappa)} G(\kappa, u_p) & ,\alpha < 0 \end{cases}$$

(A.8.8)

A.9 LOG-NORMAL DISTRIBUTION

Definition

Parameters: ξ (location), μ_y (mean), σ_y (standard deviation)

Range: $\sigma_y > 0$, $x > \xi$

If X is distributed according to a log-normal distribution, then $Y = \ln(X-\xi)$ is normally distributed. The parameters μ_y and σ_y^2 are the population mean and variance of Y.

$$f(x) = \frac{1}{(x-\xi)\sigma_y\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln(x-\xi)-\mu_y}{\sigma_y}\right)^2\right] \tag{A.9.1}$$

$$F(x) = \Phi\left(\frac{\ln(x-\xi)-\mu_y}{\sigma_y}\right) \tag{A.9.2}$$

$$x_p = \xi + \exp\left(\mu_y + \sigma_y\Phi^{-1}(p)\right) \tag{A.9.3}$$

where $\Phi(.)$ and $\Phi^{-1}(.)$ are, respectively, the cumulative distribution function and the quantile function of the standard normal distribution.

Moments

$$\mu_x = \xi + \exp\left[\mu_y + \frac{1}{2}\sigma_y^2\right] \tag{A.9.4}$$

$$\sigma_x^2 = \left(\exp\left[2\mu_y + \sigma_y^2\right]\right)\left(\exp\left(\sigma_y^2\right)-1\right) \tag{A.9.5}$$

$$\gamma_{3x} = 3\phi + \phi^3 \quad , \quad \phi = \sqrt{\exp\left(\sigma_y^2\right)-1} \tag{A.9.6}$$

L-moments

$$\lambda_{1,y} = \mu_y \tag{A.9.7}$$

$$\lambda_{2,y} = \frac{\sigma_y}{\sqrt{\pi}} \tag{A.9.8}$$

Moment estimates

If ξ is known, moment estimates of μ_y and σ_y are given by the sample mean and standard deviation of the logarithmic transformed data $\{y_i = \ln(x_i\text{-}\xi),\ i = 1,2,\ldots,n\}$.

If ξ is unknown, four different estimation methods are available. Two methods based on a lower bound quantile estimator of ξ, and two methods based on the sample moments in real space $\{x_i,\ i=1,2,\ldots,n\}$ where a bias correction of the sample skewness is adopted.

Lower bound quantile estimators

The lower bound quantile estimator of ξ proposed by Iwai [1947] is given by

$$\hat{\xi} = \frac{1}{M}\sum_{i=1}^{M}\frac{x_{(i)}x_{(n+i-1)} - x_g^2}{x_{(i)} + x_{(n+i-1)} - 2x_g} \tag{A.9.9}$$

where $x_{(n)} \le x_{(n-1)} \le \ldots \le x_{(1)}$ is the ordered sample, M is the truncated integer value of $n/10$, and $x_g = (x_1 x_2 \ldots x_n)^{1/n}$ is the geometric mean. The restriction $x_{(i)} + x_{(n+i-1)} - 2x_g > 0$ must be satisfied to obtain an estimate of ξ.

Stedinger [1980] proposed a slightly different estimator, which uses the sample median instead of the geometric mean and includes only the largest and the smallest observed values, i.e.

$$\hat{\xi} = \frac{x_{(1)}x_{(n)} - x_{med}^2}{x_{(1)} + x_{(n)} - 2x_{med}} \tag{A.9.10}$$

where x_{med} is the sample median equal to $x_{((n+1)/2)}$ for odd sample sizes, and $\frac{1}{2}(x_{(n/2)}+x_{(n/2+1)})$ for even sample sizes.

Having estimated the location parameter, estimates of μ_y and σ_y are given by the sample mean and standard deviation of the logarithmic transformed data $\{y_i = \ln(x_i - \hat{\xi}),\ i = 1,2,\ldots,n\}$.

Sample moments in real space

For estimation of the three parameters from the sample moments of $\{x_i, i=1,2,\ldots,n\}$ a bias correction of the sample skewness is adopted

$$\hat{\gamma}_3^* = (1+\beta)\hat{\gamma}_3 \tag{A.9.11}$$

Two different bias correction formulae are employed (1) the Ishihara-Takase formula, and (2) the Bobée-Robitaille formula.

In the bias correction procedure proposed by Ishihara and Takase [1957] an estimation method based on order statistics is employed. In this case the following parameterisation of the log-normal distribution is applied

$$F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{u} \exp(-t^2)dt \quad , \quad u = \kappa \ln \frac{x-\xi}{x_0 - \xi} \tag{A.9.12}$$

A reduced variate y is defined as follows

$$y = \frac{x-\xi}{x_0-\xi} = \exp\left(\frac{u}{\kappa}\right) \tag{A.9.13}$$

Since y is a linear transformation of x, the coefficient of skewness of y and x are identical. The expected value of the ordered sample $u_{(1)} \le u_{(2)} \le \ldots \le u_{(n)}$ is determined by using the Hazen plotting position

$$E\{u_{(i)}\} = \frac{1}{\sqrt{2}} \Phi^{-1}\left(\frac{i-0.5}{n}\right) \tag{A.9.14}$$

An estimate of κ can now be found by solving

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\left(y_{(i)}^{*}-\overline{y_{(i)}^{*}}\right)^{3}}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(y_{(i)}^{*}-\overline{y_{(i)}^{*}}\right)^{2}\right]^{3/2}}=\hat{\gamma}_{3x} \quad , \quad y_{(i)}^{*}=\exp\left(\frac{E\{u_{(i)}\}}{\kappa}\right) \quad , \quad \overline{y_{(i)}^{*}}=\frac{1}{n}\sum_{i=1}^{n}y_{(i)}^{*}$$

(A.9.15)

using an iterative scheme. The bias correction factor β is then given by

$$\hat{\gamma}_3^{*}=(1+\beta)\hat{\gamma}_3 \quad , \quad \beta=\frac{\hat{\gamma}_3^{*}}{\hat{\gamma}_3}-1 \tag{A.9.16}$$

where $\hat{\gamma}_3^{*}$ is obtained from

$$\hat{\gamma}_3^{*}=\frac{\exp\left(\frac{9}{4\hat{\kappa}^2}\right)-3\exp\left(\frac{5}{4\hat{\kappa}^2}\right)+2\exp\left(\frac{3}{4\hat{\kappa}^2}\right)}{\left[\exp\left(\frac{1}{\hat{\kappa}^2}\right)-\exp\left(\frac{1}{2\hat{\kappa}^2}\right)\right]^{3/2}} \tag{A.9.17}$$

The bias correction factor is shown in Fig A.9.4.

The parameter σ_y is estimated from the bias-corrected skewness estimator cf. (A.9.6) using a Newton-Raphson iteration scheme. Estimates of ξ and μ_y are subsequently obtained from (A.9.4)-(A.9.5)

The bias correction proposed by Bobée and Robitaille [1975] reads

$$\beta=\left(0.01+\frac{7.01}{n}+\frac{14.66}{n^2}\right)+\left(\frac{1.69}{n}+\frac{74.66}{n^2}\right)\hat{\gamma}_3^{3} \tag{A.9.18}$$

which is valid for $0.25 \le \gamma_3 \le 5.0$ and $20 \le n \le 90$. The bias correction factor β is shown in Fig A.9.5. If γ_3 or n fall outside the ranges of the Bobée-

Robitaille formula, the skewness is corrected using the following general bias correction

$$\hat{\gamma}_3^* = \frac{\sqrt{n(n-1)}}{n-2}\hat{\gamma}_3 \tag{A.9.19}$$

$$\hat{\mu}_y = \ln\hat{\sigma}_x - \frac{1}{2}\left[\ln\left(\exp(\hat{\sigma}_y^2)-1\right)+\hat{\sigma}_y^2\right] \quad , \quad \hat{\xi} = \hat{\mu}_x - \exp\left[\hat{\mu}_y + \frac{1}{2}\hat{\sigma}_y^2\right] \tag{A.9.20}$$

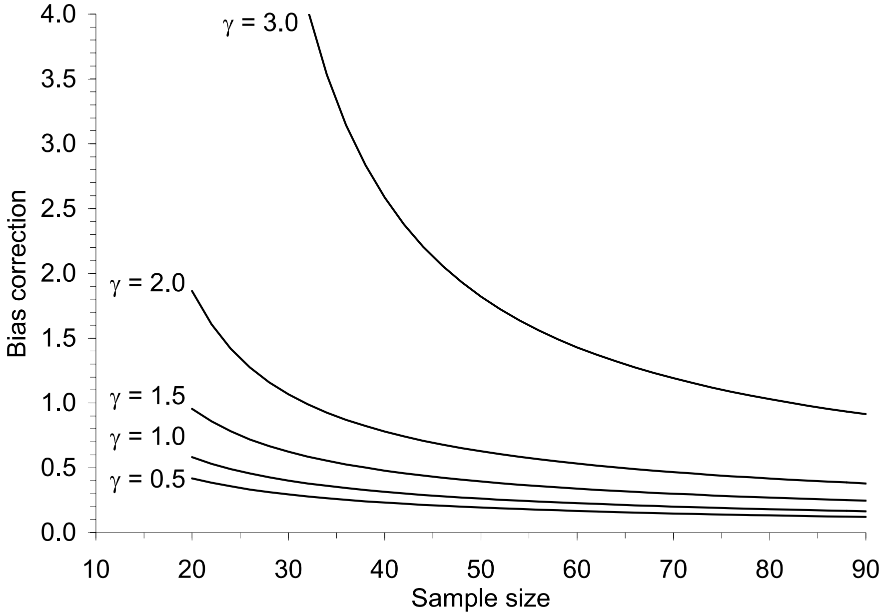

Fig A.9.4 *Bias correction factor β of the sample skewness γ for the log-normal distribution [Ishihara and Takase, 1957].*

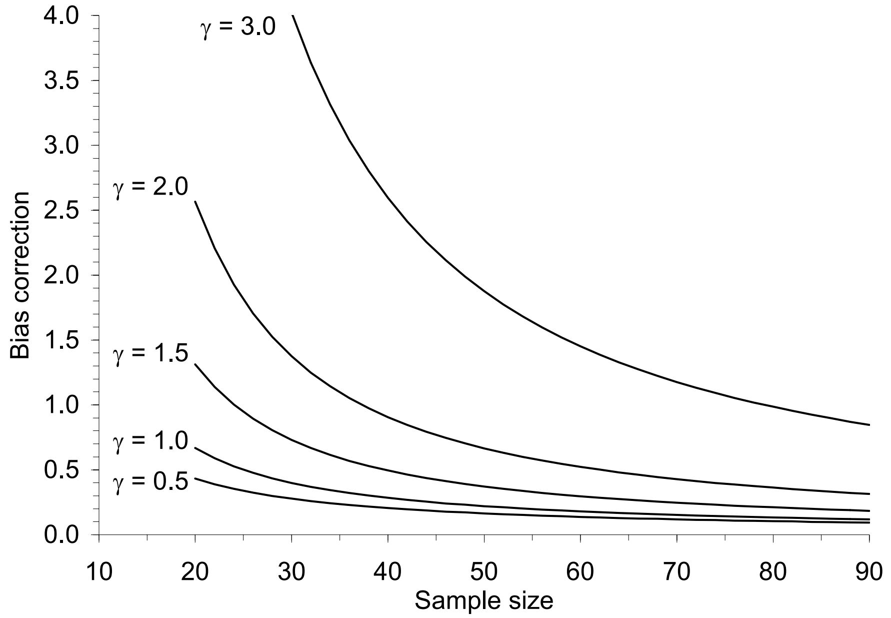

Fig A.9.5 *Bias correction factor β of the sample skewness γ for the log-normal distribution [Bobée and Robitaille, 1975].*

L-moment estimates

If ξ is known, μ_y and σ_y are estimated from the sample L-moments of the logarithmic transformed data $\{y_i = \ln(x_i - \xi),\ i = 1,2,...,n\}$.

$$\hat{\mu}_y = \hat{\lambda}_{1,y} \quad , \quad \hat{\sigma}_y = \sqrt{\pi}\,\hat{\lambda}_{2,y} \tag{A.9.21}$$

Maximum likelihood estimates

If ξ is known, maximum likelihood estimates of μ_y and σ_y are given by

$$\hat{\mu}_y = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i - \xi) \quad , \quad \hat{\sigma}_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[\ln(x_i - \xi) - \hat{\mu}_y\right]^2} \tag{A.9.22}$$

If ξ is unknown, the maximum likelihood estimate of ξ is obtained by solving

$$\frac{\partial L}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\sum_{i=1}^{n} \ln\left(\sqrt{2\pi}\sigma(x_i - \xi)\right) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln(x_i - \xi) - \mu}{\sigma}\right)^2 \right) = 0 \quad \text{(A.9.23)}$$

using a bisection iteration scheme. The parameter estimates of μ_y and σ_y are subsequently obtained from (A.9.22).

Reduced variate

$$\text{SLSC1:} \qquad u_p = \frac{\ln(x_p - \xi) - \mu}{\sigma} = \Phi^{-1}(p) \quad \text{(A.9.24)}$$

A.10 SQUARE ROOT EXPONENTIAL DISTRIBUTION

Definition

Parameters: α (scale), κ (shape)

Range: $\alpha > 0$, $\kappa > 0$, $x \geq 0$

The distribution was defined by Etoh et al. [1987].

$$f(x) = \frac{\alpha\kappa}{2} \exp\left[-\sqrt{\alpha x} - \kappa\left(1+\sqrt{\alpha x}\right)\exp\left(-\sqrt{\alpha x}\right)\right] \tag{A.10.1}$$

$$F(x) = \begin{cases} \exp\left[-\kappa\left(1+\sqrt{\alpha x}\right)\exp\left(-\sqrt{\alpha x}\right)\right] & , \quad x > 0 \\ \exp(-\kappa) & , \quad x = 0 \end{cases} \tag{A.10.2}$$

$$\left(1+\sqrt{\alpha x_p}\right)\exp\left(-\sqrt{\alpha x_p}\right) + \frac{1}{\kappa}\ln p = 0 \tag{A.10.3}$$

The square root exponential distribution is a mixed distribution with a finite probability mass placed at $x = 0$. The remaining probability is continuously distributed for $x > 0$. No explicit expression of the quantile function exists. The quantile is calculated from (A.10.3) using Newton-Raphson iteration.

Maximum likelihood estimates

The maximum likelihood estimate of α is obtained from

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{2\sqrt{\alpha}}\sum_{i=1}^{n}\sqrt{x_i} + \frac{n\sum_{i=1}^{n} x_i \exp\left(-\sqrt{\alpha x_i}\right)}{2\sum_{i=1}^{n}\left(1+\sqrt{\alpha x_i}\right)\exp\left(-\sqrt{\alpha x_i}\right)} = 0 \tag{A.10.4}$$

using Newton-Raphson iteration. The estimate of κ is subsequently found from

$$\hat{\kappa} = \frac{n}{\sum_{i=1}^{n}\left(1+\sqrt{\hat{\alpha} x_i}\right)\exp\left(-\sqrt{\hat{\alpha} x_i}\right)} \tag{A.10.5}$$

Reduced variate

$$\mathrm{SLSC1:} \qquad u_p = \sqrt{\alpha x_p} - \ln\left[\kappa\left(1+\sqrt{\alpha x_p}\right)\right] = -\ln\left[-\ln(p)\right] \qquad \text{(A.10.6)}$$

$$\mathrm{SLSC2:} \qquad u_p = \alpha x_p \quad , \quad \left(1+\sqrt{u_p}\right)\exp\left(-\sqrt{u_p}\right) = -\frac{1}{\kappa}\ln(p) \quad \text{(A.10.7)}$$

A.11 AUXILIARY FUNCTIONS

Gamma function

For calculation of the gamma function, a numerical function that calculates the logarithm of the gamma function is employed. The applied numerical method is that of Pike and Hill [1966].

Euler's psi function

Euler's psi function is the derivative of the logarithm of the gamma function

$$\psi(x) = \frac{d}{dx}\left(\ln(\Gamma(x)\right) \tag{A.11.1}$$

The applied numerical method for calculation of Euler's psi function is that of Bernardo [1976].

Incomplete gamma integral

The incomplete gamma integral is defined as

$$G(\kappa, x) = \frac{1}{\Gamma(\kappa)} \int_0^x t^{\kappa-1} \exp(-t)\,dt \tag{A.11.2}$$

The applied numerical method is that of Shea [1988].

Cumulative distribution function of standard normal distribution

The cumulative distribution function of the standard normal distribution $\Phi(.)$ can be expressed in terms of the error function *erf*(.)

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right) \tag{A.11.3}$$

For calculation of the error function the numerical method in Hart et al. [1968] based on a rational function approximation is applied.

Quantile function of standard normal distribution

The numerical method applied for calculation of the quantile of the standard normal distribution is that of Wichura [1988] which is based on a rational function approximation.